CANONICAL MODELS OF FILTERED A_∞-ALGEBRAS AND MORSE COMPLEXES

KENJI FUKAYA, YONG-GEUN OH, HIROSHI OHTA, AND KAORU ONO

In honor of Yasha Eliashberg's sixtieth birthday

1. Introduction.

In the book [6], the authors studied the moduli spaces of bordered stable maps of genus 0 with Lagrangian boundary condition in a systematic way and constructed the filtered A_∞-algebra associated to Lagrangian submanifolds. Since our construction depends on various auxiliary choices, we considered the *canonical model* of filtered A_∞-algebras, which is unique up to filtered A_∞-isomorphisms. The aim of this note is to explain the construction of the canonical model and to apply such an argument to obtain the filtered A_∞-structure to the Morse complex on the Lagrangian submanifold. The resulting filtered A_∞-operations are described by the moduli spaces of certain configurations consisting of pseudo-holomorphic curves and gradient flow lines. Note that the first named author [4] studied the quantization of Morse homotopy based on the moduli spaces of certain configurations consisting of pseudo-holomorphic discs and gradient flow lines of *multiple* Morse functions, see [13] for monotone case and Theorem A4.28 in §A 4 in [5]. Such configurations are also studied in monotone case by Buhovsky [2] and Biran and Cornea [1]. We follow Chapter 5 in [6] to explain the algebraic aspect of canonical models and use the geometric construction in Chapter 7 in [6].

We briefly review the background of our study. Floer [3] invented a new theory, which is now called Floer (co)homology for Lagrangian intersections. Very roughly speaking, it is an analog of Morse theory for the action functional on the space of paths with end points on Lagrangian submanifolds. For a transversal pair of Lagrangian submanifolds L_0, L_1, the cochain complex is generated by the intersection points of L_0 and L_1. The coboundary operator is defined by counting connecting orbits joining the intersection points. The theory was extended by the second named author [12] to the class of monotone Lagrangian submanifolds with the minimal Maslov number being at least 3. In general, however, there arise obstructions to constructing the Floer cochain complex caused by the bubbling-off of pseudo-holomorphic discs in the moduli space of connecting orbits. We started a systematic study of the moduli spaces of pseudo-holomorphic discs with Lagrangian boundary condition and formulated the obstructions in terms of the Mauer-Cartan equation on the filtered A_∞-algebra associated to the Lagrangian submanifold [6]. In order to give consistent orientations on the moduli spaces, we introduced the

K. F. is partially supported by JSPS Grant-in-Aid for Scientific Research No. 18104001, Y. O. by US NSF grant No. 0503954, H. O. by JSPS Grant-in-Aid for Scientific Research No. 19340017, and K. O. by JSPS Grant-in-Aid for Scientific Research 18340014.

notion of relative spin structure and cosidered relative spin Lagrangian submanifolds. For a relative spin pair (L_0, L_1) of Lagrangian submanifolds, we constructed a filtered A_∞-bimodule over the A_∞-algbras associated to L_0 and L_1. If each L_i, $i = 0, 1$, admits a solution b_i of the Maurer-Cartan equation, we can rectify the Floer operator to obtain a coboundary operator δ^{b_0,b_1}. Hence the Floer complex $(CF^\bullet(L_0, L_1), \delta^{b_0,b_1})$ is obtained. We also considered the case that the Lagrangian submanifolds admit solutions of the Maurer-Cartan equation modulo multiples of the fundamental class $[L_i]$ (*weak solution*). For a weak solution b_i, we assign the potential $\mathfrak{PO}(b_i)$. If $\mathfrak{PO}(b_0) = \mathfrak{PO}(b_1)$, we can construct the Floer complex $(CF(L_0, L_1), \delta^{b_0,b_1})$ deformed by b_0, b_1. This extension with the weak bounding cochains plays crutial role in our study of Floer theory on compact toric manifolds [7].

We firstly constucted the filtered A_∞-algebra mentioned above on suitable subcomplex of the singular cochain complex of L_i using systematic multi-valued perturbation of Kuranshi maps describing the moduli spaces. We briefly review these constructions in subsequent section. Thus the resulting filtered A_∞-algebra depends on various choices, i.e., the choice of the subcomplex, the choice of systematic multi-valued perturbation, etc. In order to make the construction canonical, we introduced the notion of the *canonical model*. Since the structure constants of the filtered A_∞-algebra depends on these choices, it is appropriate to work with the canonical model when we make practical computation of the structure constants. When we consider $\mathfrak{PO}(b)$ as a function on the set of weak solutions of the Maurer-Cartan equation, we call it the *potential function*. The canonical model provides an appropriate domain of the definition of the potential function. The canonical models also play a role in the convergence of a certain spectral sequence, see Chapter 6 in [6]. (In [5], we used another kind of finitely generated complex to ensure the weak finiteness property of the filtered A_∞ algebras.) It may be also worth mentioning that we rely on canonical models in some places in [6], since the degree of the ordinary cohomology is bounded, though the degree of the singular complex is not bounded above.

We also developed an algebraic theory for filtered A_∞-algebras, bimodules, in particular, the homotopy theory of the filtered A_∞-algebras, bimodules and proved that the homotopy type of the resulting algebraic object does not depend on such choices. We can also reduce the filtered A_∞-structure to appropriate free subcomplexes of the original complex. In particular, if we work over the ground coefficient field, we obtain the filtered A_∞-structure on the *classical* (co)homology of the complex. In this note, we review the construction of the canonical models of filtered A_∞-algebras and filtered A_∞-bimodules and explain its implication in a geometric setting.

In section 5, we induce the filtered A_∞-algebra structure on Morse complex based on the argument in the construction of canonical models. We choose a Morse function f on L, which is adapted to a triangulation of L (see section 5).

Theorem 5.1 *Let L be a relatively spin Lagrangian submanifold in a closed symplectic manifold (M, ω) and f a Morse function on L as above. Then Morse complex $CM^*(f) \otimes \Lambda_{nov}$ carries a structure of a filtered A_∞-algebra, which is homotopy equivalent to the filtered A_∞-algebra associated to L constructed in* [6].

This note is based on the lecture at Yashafest and we thank the organizers for the invitation.

2. Flitered A_∞-algebras, bimodules

In this subsection, we recall the definition of (filtered) A_∞-algebras, bimodules, homomorphisms and prepare necessary notations. Then we explain the notion of homotopy between (filtered) A_∞-homomorphisms. In fact, the notion of homotopy between A_∞-homomorphisms can be found in the literature, e.g., [14]. For differential graded algebras, homotopy theory was studied in rational homotopy theory, see [16], [9]. In order to make clear the relation among such notions, we introduced the notion of the model of $[0,1]\times\overline{C}$ ($[0,1]\times C$) for a (filtered) A_∞-algebra C ($\overline{C}$) and defined the notion of homotopy using such a model.

2.1) Unfiltered A_∞-algebras, homomorphisms, bimodules, bimodule homomorphisms.

Let R be a commutative ring, e.g., $\mathbb{Z}$, $\mathbb{Q}$. Let $\overline{C}^\bullet$ be a cochain complex over R. We assume that $\overline{C}^k=0$ for $k<0$. Denote by $\overline{\mathfrak{m}}_1$ its differential. Set $\overline{C}[1]^k=\overline{C}^{k+1}$ and denote the shifted degree by $\deg' x=\deg x-1$, where deg is the original degree of $\overline{C}^\bullet$. In this section, we use only shifted degrees. Consider a series of k-ary operations, $k=1,2,\ldots,$

$$\overline{\mathfrak{m}}_k:(\overline{C}[1]^\bullet)^{\otimes k}\to\overline{C}[1]^\bullet$$

of degree 1 with respect to the shifted degrees.

Before giving the definition of A_∞-algebras, we explain the case of differential graded algebras. Let $(\overline{C}^\bullet,d,\cdot)$ be a differential graded algebra. Define $\overline{\mathfrak{m}}_1(x)=(-1)^{\deg x}da$ and $\overline{\mathfrak{m}}_2(x\otimes y)=(-1)^{\deg x\cdot(\deg y+1)}x\cdot y$. Then we find that

$$\begin{aligned}
&\overline{\mathfrak{m}}_1\circ\overline{\mathfrak{m}}_1(x)=0,\\
&\overline{\mathfrak{m}}_1\circ\overline{\mathfrak{m}}_2(x\otimes y)+\overline{\mathfrak{m}}_2(\overline{\mathfrak{m}}_1(x_1)\otimes x_2)+(-1)^{\deg' x_1}\overline{\mathfrak{m}}_2(x_1\otimes\overline{\mathfrak{m}}_1(x_2))=0,\\
&\overline{\mathfrak{m}}_2(\overline{\mathfrak{m}}_2(x_1\otimes x_2)\otimes x_3)+(-1)^{\deg' x_1}\overline{\mathfrak{m}}_2(x_1\otimes\overline{\mathfrak{m}}_2(x_2\otimes x_3))=0,
\end{aligned}$$

which follow from the facts that d is a differential, the multiplication and the differential d satisfies Leibniz' rule and the multiplication is associative.

There are some geometric situations where multiplicative structures are defined but not exactly associative. A typical example is the composition in based loop spaces. In fact, Stasheff [15] introduced the notion of A_∞-structure on topological spaces in order to characterize the homotopy types of based loop spaces. He also defined the A_∞-structure in algebraic setting. For instance, a multiplicative structure is said to be associative up to homotopy, if there exists $\overline{\mathfrak{m}}_3:(\overline{C}[1]^\bullet)^{\otimes 3}\to\overline{C}[1]^\bullet$ such that

$$\begin{aligned}
&\overline{\mathfrak{m}}_2(\overline{\mathfrak{m}}_2(x_1\otimes x_2)\otimes x_3)+(-1)^{\deg' x_1}\overline{\mathfrak{m}}_2(x_1\otimes\overline{\mathfrak{m}}_2(x_2\otimes x_3))\\
+&\overline{\mathfrak{m}}_1\circ\overline{\mathfrak{m}}_3(x_1\otimes x_2\otimes x_3)+\overline{\mathfrak{m}}_3(\overline{\mathfrak{m}}_1(x_1)\otimes x_2\otimes x_3)\\
+&(-1)^{\deg' x_1}\overline{\mathfrak{m}}_3(x_1\otimes\overline{\mathfrak{m}}_1(x_2)\otimes x_3)+(-1)^{\deg' x_1+\deg' x_2}\overline{\mathfrak{m}}_3(x_1\otimes x_2\otimes\overline{\mathfrak{m}}_1(x_3))\\
=&0.
\end{aligned}$$

Note that it coincides with the relation corresponding to the associativity, if $\overline{\mathfrak{m}}_3 = 0$. We can continue higher homotopies in a similar way:

$$\sum_{k_1+k_2=k+1} \sum_i (-1)^{\sum_{j=1}^{i-1} \deg' x_j} \overline{\mathfrak{m}}_{k_1}(x_1, \dots, \overline{\mathfrak{m}}_{k_2}(x_i, \dots, x_{i+k_2-1}), \dots, x_k) = 0.$$

Here k_1, k_2 are positive integers. For a concise description of relations among higher homotopies, we introduce the bar complex of $\overline{C}^\bullet$, which is defined by

$$B(\overline{C}[1]^\bullet) = \bigoplus_{k=0}^{\infty} B_k(\overline{C}[1]^\bullet), \quad B_k(\overline{C}[1]^\bullet) = \bigoplus_{m_1,\dots,m_k} \overline{C}[1]^{m_1} \otimes \cdots \otimes \overline{C}[1]^{m_k},$$

which we consider as a tensor coalgebra. The comultiplication is given by

$$\Delta(x_1 \otimes \cdots \otimes x_k) = \sum_{i=0}^{k} (x_1 \otimes \cdots \otimes x_i) \otimes (x_{i+1} \otimes \cdots \otimes x_k),$$

where $x_1 \otimes \cdots \otimes x_i, x_{i+1} \otimes \cdots \otimes x_k \in B(\overline{C}[1]^\bullet)$ and the former with $i = 0$ and the latter with $i = k$ are understood as $1 \in B_0(\overline{C}[1]^\bullet)$. Extend $\overline{\mathfrak{m}}_k$ to the graded coderivation $\widehat{\overline{\mathfrak{m}}}_k$ on $B(\overline{C}[1]^\bullet)$. Namely,

$$\begin{aligned}
&\widehat{\overline{\mathfrak{m}}}_k(x_1 \otimes \cdots \otimes x_N) \\
&= \sum_{i=1}^{N-k+1} (-1)^{\sum_{j=1}^{i-1} \deg' x_j} x_1 \otimes \cdots \otimes x_{i-1} \otimes \overline{\mathfrak{m}}_k(x_i \otimes \cdots \otimes x_{i+k-1}) \otimes x_{i+k} \otimes \dots \\
&\qquad \cdots \otimes x_N.
\end{aligned}$$

We call $(\overline{C}^\bullet, \{\overline{\mathfrak{m}}_k\})$ an A_∞-algebra, if

$$\widehat{\overline{d}} = \sum_k \widehat{\overline{\mathfrak{m}}}_k : B(\overline{C}[1]^\bullet) \to B(\overline{C}[1]^\bullet)$$

satisfies $\widehat{\overline{d}} \circ \widehat{\overline{d}} = 0$. In the case that $\overline{\mathfrak{m}}_k = 0$ for $k > 2$, this condition is equivalent to the notion of differential graded algebras.

For a collection $\{\overline{\mathfrak{f}}_k : B_k(\overline{C}[1]^\bullet) \to \overline{C}'[1]^\bullet\}_{k=1}^{\infty}$ of degree 0, we extend it to a homomorphism as tensor coalgebras

$$\widehat{\overline{f}}(x_1 \otimes \cdots \otimes x_k) = \sum_{k_1+\cdots+k_n=k} \overline{\mathfrak{f}}_{k_1}(x_1 \otimes \cdots \otimes x_{k_1}) \otimes \cdots \otimes \overline{\mathfrak{f}}_{k_n}(x_{k+1-k_n} \otimes \cdots \otimes x_k).$$

We call $\{\overline{\mathfrak{f}}_k\}$ an A_∞-homomorphism, if $\widehat{\overline{f}}$ satisfies $\widehat{\overline{d}}_{\overline{C}'} \circ \widehat{\overline{f}} = \widehat{\overline{f}} \circ \widehat{\overline{d}}_{\overline{C}}$.

In terms of the components $\overline{\mathfrak{m}}_k$'s and $\overline{\mathfrak{f}}_k$'s, this is equivalent to

$$\begin{aligned}
&\sum_{i_1+\cdots+i_k=n} \overline{\mathfrak{m}}_k(\overline{\mathfrak{f}}_{i_1}(x_1 \otimes \cdots \otimes x_{i_1}) \otimes \cdots \otimes \overline{\mathfrak{f}}_{i_k}(x_{i_1+\cdots+i_{k-1}+1} \otimes \cdots \otimes x_n)) \\
&= \sum_{j_1+\cdots+j_\ell=n} \sum_{p=1}^{\ell} (-1)^{\sum_{i=1}^{j_1+\dots j_{p-1}} \deg' x_i} \overline{\mathfrak{f}}_{j_1}(x_1 \otimes \cdots \otimes x_{j_1}) \otimes \cdots \otimes \\
&\quad \overline{\mathfrak{m}}_{j_p}(x_{j_1+\cdots+j_{p-1}+1} \otimes \cdots \otimes x_{j_1+\cdots+j_p}) \otimes \cdots \otimes \overline{\mathfrak{f}}_{j_\ell}(x_{j_1+\dots j_{\ell-1}+1} \otimes \cdots \otimes x_n)
\end{aligned}$$

Let $(\overline{C}_i^\bullet, \{\overline{\mathfrak{m}}_k^{(i)}\})$, $i=0,1$, be A_∞-algebras, $\overline{D}^\bullet$ a graded module and $\overline{\mathfrak{n}}_{k_1,k_0} : B_{k_1}(\overline{C}_1[1]^\bullet)\otimes\overline{D}[1]^\bullet\otimes B_{k_0}(\overline{C}_0[1]^\bullet)\to\overline{D}[1]^\bullet$ homomorphisms of degree 1. We call $(\overline{D}^\bullet, \{\overline{\mathfrak{n}}_{k_1,k_0}\})$ an A_∞-bimodule, if

$$\widehat{\overline{d}}_{\overline{\mathfrak{n}}}\circ\widehat{\overline{d}}_{\overline{\mathfrak{n}}}=0,$$

where $\widehat{\overline{d}}_{\overline{\mathfrak{n}}}$ is defined on $B(\overline{C}_1[1]^\bullet)\otimes\overline{D}[1]^\bullet\otimes B(\overline{C}_0[1]^\bullet)$ as follows:

$$\begin{aligned}
&\widehat{\overline{d}}_{\overline{\mathfrak{n}}}(x_{1,1}\otimes\cdots\otimes x_{1,k_0}\otimes y\otimes x_{0,1}\otimes\cdots\otimes x_{0,k_0})\\
=&\widehat{\overline{d}}^{(1)}(x_{1,1}\otimes\cdots\otimes x_{1,k_1})\otimes y\otimes x_{0,1}\otimes\cdots\otimes x_{0,k_0}\\
&+\sum_{k_1'\le k_1,k_0'\le k_0}(-1)^{\sum_{i=1}^{k_1-k_1'}\deg' x_i}x_{1,1}\otimes\dots x_{k_1-k_1'}\\
&\otimes\overline{\mathfrak{n}}_{k_1',k_0'}(x_{1,k_1-k_1'+1}\otimes\dots x_{k_1}\otimes y\otimes x_{0,1}\otimes\cdots\otimes x_{0,k_0'})\otimes x_{0,k_0+1}\otimes\cdots\otimes x_{0,k_0}\\
&+(-1)^{\sum_{i=1}^{k_1}\deg' x_i+\deg' y}x_{1,1}\otimes\dots x_{1,k_1}\otimes y\otimes\widehat{\overline{d}}^{(0)}(x_{0,1}\otimes\cdots\otimes x_{0,k_0}).
\end{aligned}$$

The condition for $(\overline{D}, \{\overline{\mathfrak{n}}_{k_1,k_0}\})$ to be an A_∞-bimodules over $\overline{C}_i$, $i=0,1$ is equivalent to the identity

$$\begin{aligned}
&\overline{\mathfrak{n}}_{*,*}(\widehat{\overline{d}}^{(1)}(x_{1,1}\otimes\cdots\otimes x_{1,k_1})\otimes y\otimes x_{0,1}\otimes\cdots\otimes x_{0,k_0})\\
&+\sum_{k_1'\le k_1,k_0'\le k_0}(-1)^{\sum_{i=1}^{k_1-k_1'}\deg' x_i}\overline{\mathfrak{n}}_{k_1-k_1',k_0-k_0'}(x_{1,1}\otimes\cdots\otimes x_{k_1-k_1'}\otimes\\
&\overline{\mathfrak{n}}_{k_1',k_0'}(x_{1,k_1-k_1'+1}\otimes\cdots\otimes x_{1,k_1}\otimes y\otimes x_{0,1}\otimes x_{0,k_0'})\otimes x_{0,k_0'+1}\otimes\cdots\otimes x_{0,k_0})\\
&+(-1)^{\sum_{i=1}^{k_1}\deg' x_i+\deg' y}\overline{\mathfrak{n}}_{*,*}(x_{1,1}\otimes\cdots\otimes x_{1,k_1}\otimes y\otimes\widehat{\overline{d}}^{(0)}(x_{0,1}\otimes\cdots\otimes x_{0,k_0}))\\
&=0.
\end{aligned}$$

Here $\overline{\mathfrak{n}}_{*,*} : B(\overline{C}_1[1]^\bullet)\otimes\overline{D}[1]\otimes B(\overline{C}_0[1]^\bullet)\to\overline{D}[1]$ is defined to be $\overline{\mathfrak{n}}_{k_1,k_0}$ on $B_{k_1}(\overline{C}_1[1]^\bullet)\otimes\overline{D}[1]\otimes B_{k_0}(\overline{C}_0[1]^\bullet)$.

Let $\{\overline{\mathfrak{f}}_k^{(i)} : \overline{C}_i[1]^\bullet\to\overline{C}_i'[1]^\bullet\}$, $i=0,1$, be A_∞-homomorphisms and $\overline{D}^\bullet$, resp. $\overline{D'}^\bullet$ an A_∞-bimodules over $\overline{C}_i$, resp. $\overline{C}_i'$, $i=0,1$. For a collection $\{\overline{\phi}_{k_1,k_0}\}$: $B_{k_1}(\overline{C}_1[1]^\bullet)\otimes\overline{D}[1]^\bullet\otimes B_{k_0}(\overline{C}_0[1]^\bullet)\to\overline{D}'[1]^\bullet$ of degree 0, we define

$$\widehat{\overline{\phi}} : B(\overline{C}_1[1]^\bullet)\otimes\overline{D}[1]^\bullet\otimes B(\overline{C}_0[1]^\bullet)\to B(\overline{C}_1'[1]^\bullet)\otimes\overline{D}'[1]^\bullet\otimes B(\overline{C}_0'[1]^\bullet)$$

as the homomorphism determined by $\{\overline{\mathfrak{f}}_k^{(i)}\}$, $i=0,1$ and $\{\overline{\phi}_{k_1,k_0}\}$. We call $\{\overline{\phi}_{k_1,k_0}\}$ a homomorphism of A_∞-bimodules, if

$$\widehat{\overline{d}}_{\overline{\mathfrak{n}}'}\circ\widehat{\overline{\phi}}=\widehat{\overline{\phi}}\circ\widehat{\overline{d}}_{\overline{\mathfrak{n}}}.$$

2.2) The universal Novikov ring and the energy filtration.

To explain the notion of filtered A_∞-algebras, we introduce the universal Novikov ring. Let e and T be formal variables of degree 2 and 0, respectively. Set

$$\Lambda_{nov} = \{\sum_i a_i e^{\mu_i} T^{\lambda_i} \mid a_i \in R,\ \mu_i \in \mathbb{Z},\ \lambda_i \in \mathbb{R},\ \lambda_i \to +\infty (i \to +\infty)\}$$

$$\Lambda_{0,nov} = \{\sum_i a_i e^{\mu_i} T^{\lambda_i} \in \Lambda_{nov} \mid \lambda_i \geq 0\}.$$

Set

$$C^\bullet = \{\sum c_i e^{\mu_i} T^{\lambda_i} | c_i \in \overline{C}^\bullet, \mu_i \in \mathbb{Z}, \lambda_i \in \mathbb{R}, \lambda_i \to +\infty (i \to +\infty)\},$$

which is the completion of the graded tensor product $\overline{C}^\bullet \otimes_R \Lambda_{0,nov}$ with respect to the energy filtration given below. We define the filtration defined by

$$F^\lambda C^\bullet = \{\sum_i x_i e^{\mu_i} T^{\lambda_i} \in C^\bullet \mid x_i \in \overline{C}^\bullet, \lambda_i \geq \lambda\}$$

on $C^\bullet$ and denote by $F^\lambda(C[1]^{m_1} \otimes \cdots \otimes C[1]^{m_k})$ the submodule of $\mathbb{C}[1]^{m_1} \otimes \cdots \otimes C[1]^{m_k}$ spanned by

$$F^{\lambda_1}(C[1]^{m_1}) \otimes \cdots \otimes F^{\lambda_k}(C[1]^{m_k}), \quad \sum_{i=1}^k \lambda_i = \lambda.$$

Define the bar complex of $C[1]^\bullet$ by the completion with respect to the energy filtration and denote it by $B_k(C[1]^\bullet)$.

2.3) Filtered case and G-gapped conditions.

Consider the k-ary operations, $k = 0, 1, 2, \dots,$

$$\mathfrak{m}_k : B_k(C[1]^\bullet) \to C[1]^\bullet$$

such that

$$\mathfrak{m}_k(F^{\lambda_1} C[1]^\bullet \otimes \cdots \otimes F^{\lambda_k} C[1]^\bullet) \subset F^{\lambda_1 + \cdots \lambda_k} C[1]^\bullet$$

and

$$\mathfrak{m}_0(1) \in F^{\lambda'} C[1]^\bullet \text{ for some } \lambda' > 0.$$

We used the induction on the energy level in various arguments in [6], see also section 3 in this note. For such purposes, we introduced the G-gapped condition, which we assume from now on, as follows. Note that the G-gapped condition follows from Gromov's compactness theorem in the case of symplectic Floer theory. Let $G \subset \mathbb{R}_{\geq 0} \times 2\mathbb{Z}$ be a monoid such that $\mathrm{pr}_1^{-1}([0, c])$ is finite for any $c \geq 0$ and

$$\mathrm{pr}_1^{-1}(0) = \{\mathbf{0} = (0, 0)\}.$$

Here pr_i is the projection to the i-th factor, $i = 1, 2$. The filtered A_∞-algebra is said to be G-gapped, if there exist

$$\mathfrak{m}_{k,\beta_i} : B_k(\overline{C}[1]^\bullet) \to \overline{C}[1]^\bullet$$

for $\beta_i = (\lambda_i, \mu_i) \in G \subset \mathbb{R}_{\geq 0} \times 2\mathbb{Z}$ such that $\mathfrak{m}_{0,\mathbf{0}} = 0$ and

$$\mathfrak{m}_k = \sum_i T^{\lambda_i} e^{\mu_i/2} \mathfrak{m}_{k,\beta_i}.$$

Extend $\mathfrak{m}_k$ to the graded coderivation $\widehat{\mathfrak{m}}_k$ on $B(C[1]^\bullet)$. We call $(C^\bullet, \{\mathfrak{m}_k\})$ a filtered A_∞-algebra, if

$$\widehat{d} = \sum_k \widehat{\mathfrak{m}}_k : B(C[1]^\bullet) \to B(C[1]^\bullet)$$

satisfies $\widehat{d} \circ \widehat{d} = 0$. In other words,

$$\sum_{k_1+k_2=k+1} \sum_i (-1)^{\sum_{j=1}^{i-1} \deg' x_j} \mathfrak{m}_{k_1}(x_1, \ldots, \mathfrak{m}_{k_2}(x_i, \ldots, x_{i+k_2-1}), \ldots, x_k) = 0.$$

Here k_1 is a positive integer and k_2 is a non-negative integers. When $k_2 = 0$, $\mathfrak{m}_{k_2}(x_i, \ldots, x_{i+k_2-1})$ is understood as $\mathfrak{m}_0(1)$.

For a filtered A_∞-algebra $(C^\bullet, \{\mathfrak{m}_k\})$, set $\overline{\mathfrak{m}}_k = \mathfrak{m}_{k,\mathbf{0}}$, $\mathbf{0} = (0,0) \in \mathbb{R}_{\geq 0} \times 2\mathbb{Z}$. Then $(\overline{C}^\bullet, \{\overline{\mathfrak{m}}_k\})$ is an A_∞-algebra. We call $(C^\bullet, \{\mathfrak{m}_k\})$ a deformation of $(\overline{C}^\bullet, \{\overline{\mathfrak{m}}_k\})$.

Note that $\mathfrak{m}_1 \circ \mathfrak{m}_1$ may not be zero and we have

$$\mathfrak{m}_1 \circ \mathfrak{m}_1(x) + \mathfrak{m}_2(\mathfrak{m}_0(1), x) + (-1)^{\deg' x} \mathfrak{m}_2(x, \mathfrak{m}_0(1)) = 0.$$

We set

$$e^b = 1 + b + b \otimes b + b \otimes b \otimes b + \ldots,$$

for $b \in \mathcal{F}^\lambda(C[1]^0)$ with $\lambda > 0$ and consider the *Maurer-Cartan* equation:

$$\widehat{d}(e^b) = 0,$$

which is equivalent to

$$\mathfrak{m}_0(1) + \mathfrak{m}_1(b) + \mathfrak{m}_2(b,b) + \mathfrak{m}_3(b,b,b) + \cdots = 0.$$

For a given b, we define a coalgebra homomorphism

$$\Phi^b(x_1 \otimes x_2 \otimes \cdots \otimes x_k) = e^b \otimes x_1 \otimes e^b \otimes x_2 \otimes e^b \otimes \cdots \otimes e^b \otimes x_k \otimes e^b.$$

Then define

$$\mathfrak{m}_k^b(x_1 \otimes \cdots \otimes x_k) = \mathfrak{m}_* \circ \Phi^b(x_1 \otimes \cdots \otimes x_k),$$

where $\mathfrak{m}_* : B(C[1]^\bullet) \to C[1]^\bullet$ is defined by $\mathfrak{m}_*|_{B_k(C[1]^\bullet)} = \mathfrak{m}_k$. Then, for a solution b of the Maurer-Cartan equation, we find that $\mathfrak{m}_0^b(1) = 0$, hence $\mathfrak{m}_1^b \circ \mathfrak{m}_1^b = 0$. Namely, the original $\mathfrak{m}_1$ is rectified to a coboundary operator $\mathfrak{m}_1^b$ using a solution of the Maurer-Cartan equation, which we also call a bounding cochain.

For a collection $\{\mathfrak{f}_k : B_k(C[1]^\bullet) \to C'[1]^\bullet\}_{k=0}^\infty$ of degree 0, we define

$$\widehat{\mathfrak{f}}(x_1 \otimes \cdots \otimes x_k) = \sum_{k_1 + \cdots k_n = k} \mathfrak{f}_{k_1}(x_1 \otimes \cdots \otimes x_{k_1}) \otimes \cdots \otimes \mathfrak{f}_{k_n}(x_{k+1-k_n} \otimes \cdots \otimes x_k),$$

for $k > 0$ and

$$\widehat{\mathfrak{f}}(1) = 1 + \mathfrak{f}_0(1) + \mathfrak{f}_0(1) \otimes \mathfrak{f}_0(1) + \cdots,$$

where $1 \in \Lambda_{0,nov} = B_0(C[1]^\bullet)$. We assume the G-gapped condition, i.e., there exist

$$\mathfrak{f}_{k,\beta_i} : B_k(\overline{C}[1]^\bullet) \to \overline{C}'[1]^\bullet$$

for $\beta_i = (\lambda_i, \mu_i) \in G$ with $\lambda_i \to +\infty$ as $i \to +\infty$ such that

$$\mathfrak{f}_k = \sum_i T^{\lambda_i} e^{\mu_i/2} \mathfrak{f}_{k,\beta_i}.$$

In particular, $\widehat{\mathfrak{f}}$ preserves the energy filtration. Namely,

$$\widehat{\mathfrak{f}}(F^\lambda B(C[1]^\bullet)) \subset F^\lambda C'[1]^\bullet,$$

where $\{F^\lambda B(C[1]^\bullet)\}$ is the filtration derived from the filtration F^λ on $C[1]^\bullet$. We call $\{\mathfrak{f}_k\}$ a G-gapped filtered A_∞-homomorphism, if $\widehat{d}'_C \circ \widehat{\mathfrak{f}} = \widehat{\mathfrak{f}} \circ \widehat{d}_C$. When we do not specify the monoid G, we call gapped filtered A_∞-algebras, gapped filtered A_∞-homomorphisms, etc.

2.4) Homotopy theory.

In [6], we introduced the notion of models of $[0,1] \times C^{\bullet}$ (Definition 15.1) and gave two constructions. Using this notion, we developed the homotopy theory of filtered A_∞-algebras and filtered-A_∞ bimodules. Our formulation has an advantage to clarify equivalence of various definitions of homotopy of A_∞ algebras appearing in the literature even for the unfiltered cases.

Let C be the completion of $\overline{C} \otimes \Lambda_0$, which is a filtered A_∞-algebra.

Definition 2.1. *Let $\mathfrak{C}$ be the completion of $\overline{\mathfrak{C}} \otimes \Lambda_0$, which is a filtered A_∞-algebra together with filtered A_∞-homomorphisms.*

$$\mathrm{Incl} : C \to \mathfrak{C},\ \mathrm{Eval}_{s=i} : \mathfrak{C} \to C, i = 0, 1.$$

We call $\mathfrak{C}$ a model of $[0,1] \times C$, if the following conditions are satisfied:

- $\mathrm{Incl}_{k,\beta}$ *and* $\mathrm{Eval}_{s=i}, i = 0, 1$ *are zero unless* $(k, \beta) = (1, \beta_0)$.
- $\mathrm{Incl}_{1,\beta_0}, (\mathrm{Eval}_{s=0})_{1,\beta_0}$ *are cochain homotopy equivalences between* $\overline{C}$ *and* $\overline{\mathfrak{C}}$.
- $\mathrm{Eval}_{s=0} \circ \mathrm{Incl} = \mathrm{Eval}_{s=1} \circ \mathrm{Incl} = \mathrm{id}$.
- $\mathrm{Eval}_{s=0} \oplus \mathrm{Eval}_{s=1} : \mathfrak{C} \to C \oplus C$ *is surjective.*

We quote here one of constructions of the models of $[0,1] \times C$ for reader's convenience.

Set

$$C^{[0,1]} = C \oplus C[-1] \oplus C,$$

and define $\mathfrak{I}_0, \mathfrak{I}_1 : C \to C^{[0,1]}$ of degree 0 and $\mathfrak{I} : C \to C^{[0,1]}$ of degree 1 by

$$\mathfrak{I}_0(x) = (x, 0, 0), \mathfrak{I}_1(x) = (0, 0, x), \mathfrak{I}_1(x) = (0, x, 0).$$

We extend $\mathfrak{I}_0, \mathfrak{I}_1$ to $B(C[1]) \to B(C^{[0,1]}[1])$ and denote them by the same symbol. Define

$$\begin{aligned}(\mathrm{Eval}_{s=0})_1(x, y, z) = x,\ (\mathrm{Eval}_{s=1})_1(x, y, z) = z,\\ (\mathrm{Incl})_1(x) = \mathfrak{I}_0(x) + \mathfrak{I}_1(x) = (x, 0, x).\end{aligned}$$

We define the filtered A_∞-structure $\{\mathfrak{M}_k\}$.

For $\mathfrak{M}_0$, $\mathfrak{M}_1$, we set

$$\begin{aligned}\mathfrak{M}_0(1) &= (\mathrm{Incl})_1(\mathfrak{m}_0(1)),\\ \mathfrak{M}_1(\mathfrak{I}_0(x)) &= \mathfrak{I}_0(\mathfrak{m}_1(x)) + (-1)^{\deg' x}\mathfrak{I}(x),\\ \mathfrak{M}_1(\mathfrak{I}_1(x)) &= \mathfrak{I}_1(\mathfrak{m}_1(x)) - (-1)^{\deg' x}\mathfrak{I}(x),\\ \mathfrak{M}_1(\mathfrak{I}(x)) &= \mathfrak{I}(\mathfrak{m}_1(x)).\end{aligned}$$

We define $\mathfrak{M}_k$, $k \geq 2$ as follows. For $\mathbf{x} \in B_k(C[1])$, $y \in C$ and $\mathbf{z} \in B_\ell(C[1])$, we set

$$\mathfrak{M}_{k+\ell+1}(\mathfrak{I}_0(\mathbf{x}), \mathfrak{I}(y), \mathfrak{I}_1(\mathbf{z})) = (-1)^{\deg' \mathbf{z}}\mathfrak{I}(\mathfrak{m}_{k+\ell+1}(\mathbf{x}, y, \mathbf{z}),$$

and

$$\mathfrak{M}_k(\mathfrak{I}_0(\mathbf{x})) = \mathfrak{I}_0(\mathfrak{m}_k(\mathbf{x})), \mathfrak{M}_\ell(\mathfrak{I}_1(\mathbf{z})) = \mathfrak{I}_1(\mathfrak{m}_\ell(\mathbf{z})), \quad \text{for } k, \ell \geq 2.$$

Here the order of $\mathfrak{I}_0(\mathbf{x}), \mathfrak{I}(y), \mathfrak{I}_1(\mathbf{z})$ is important. We define operators $\mathfrak{M}_k$ on $C^{[0,1]}$ other than those defined above to be zero.

Models of $[0,1] \times C$ are not unique, but we proved the following:

Theorem 2.1 (Theorem 15.34 in [6]). *Let C_1, C_2 be gapped filtered A_∞-algebras and $\mathfrak{C}_1, \mathfrak{C}_2$ any models for $[0,1]\times C_1, [0,1]\times C_2$, respectively. Let $\mathfrak{f} : C_1 \to C_2$ be a gapped filtered A_∞-homomorphism. Then there exists a gapped filtered A_∞-homomorphism $\mathfrak{F} : \mathfrak{C}_1 \to \mathfrak{C}_2$ such that*

$$\mathrm{Eval}_{s=s_0} \circ \mathfrak{F} = \mathfrak{f} \circ \mathrm{Eval}_{s=s_0}, \;\; s_0 = 0, 1$$

and

$$\mathrm{Incl} \circ \mathfrak{f} = \mathfrak{F} \circ \mathrm{Incl}.$$

We define two filtered A_∞-homomorphisms $\mathfrak{f}_i : C_1 \to C_2, i = 0, 1$ are homotopic, if there is a model $\mathfrak{C}_2$ of C_2 and a filtered A_∞-homomorphism $\mathfrak{F} : C_1 \to \mathfrak{C}_2$ such that $\mathfrak{f}_i = \mathrm{Eval}_{s=i} \circ \mathfrak{F}$. Although the definition literally depends on the choice of the model $\mathfrak{C}_2$, we can show that the notion of homotopy between $\mathfrak{f}_i$ does not depend on the choice of the model and the homotopy is, in fact, an equivalence relation, see Chapter 4 in [6].

Note that the notion of homotopy between A_∞-homomorphisms in the unfiltered case appeared in literaturure, e.g., [14]. By taking a suitable model, we can find that our definition above coincides with such a definition. It also implies that various definitions which appear in the literature are equivalent to one another. We think that the notion of models clarifies arguments and is also useful when we consider the gauge equivalence between solutions of the Maurer-Cartan equation. Namely, two solutions b, b' of the Maurer-Cartan equation is gauge equivalent, if there is a model $\mathfrak{C}$ of $[0,1]\times C$ and a solution $\widetilde{b}$ of the Maurer-Cartan eqution on $\mathfrak{C}$ such that $\mathrm{Eval}_{s=0}(\widetilde{b}) = b$ and $\mathrm{Eval}_{s=1}(\widetilde{b}) = b'$. For details, see section 16 in [6].

Among other things, we proved the Whitehead type theorem as follows. An A_∞-homomorphism $\{\overline{\mathfrak{f}}_k\}$ from $\overline{C}^\bullet$ to $\overline{C}^{'\bullet}$ is called a weak homotopy equivalence, if $\overline{f}_1 : \overline{C}^\bullet \to \overline{C}^{'\bullet}$ is a cochain homotopy equivalence between $\overline{\mathfrak{m}}_1$-complexes. A filtered A_∞-homomorphism $\{\mathfrak{f}_k\}$ from $C[1]^\bullet$ to $C'[1]^\bullet$ is called a weak homotopy equivalence, if $\overline{\mathfrak{f}}_1 = \mathfrak{f}_{1,\mathbf{0}}$ is a cochain homotopy equivalence between $\overline{\mathfrak{m}}_1 = \mathfrak{m}_{1,\mathbf{0}}$-complexes.

Theorem 2.2 (Theorem 15.45 in [6]). *(1) A weak homotopy equivalence of A_∞-algebras is a homotopy equivalence.*
(2) A gapped weak homotopy equivalence between gapped filtered A_∞-algebras is a homotopy equivalence. The homotopy inverse of a strict weak homotopy equivalence can be taken to be strict.

Note that the above theorem does not hold in the realm of differential graded algebras. The notion of A_∞-homomorphism is much wider than that of homomorphisms as differential graded algebras.

2.5) Filtered A_∞-bimodules.

Let $(C_i^\bullet, \{\mathfrak{m}_k^{(i)}\})$, $i = 0, 1$, be filtered A_∞-algebras and $\overline{D}^\bullet$ a graded module. Write

$$D[1]^\bullet = \overline{D}[1]^\bullet \otimes \Lambda_{0,nov}$$

and

$$\widetilde{D}[1]^\bullet = \overline{D}[1]^\bullet \otimes \Lambda_{nov}.$$

Let $\mathfrak{n}_{k_1,k_0} : B_{k_1}(C_1[1]^\bullet) \otimes D[1]^\bullet \otimes B_{k_0}(C_0[1]^\bullet) \to D[1]^\bullet$, $k_1, k_0 = 0, 1, 2, \ldots,$ be $\Lambda_{0,nov}$-module homomorphisms of degree 1. We also denote its extension to $\widetilde{D}[1]^\bullet$

by the same symbol $\mathfrak{n}_{k_1,k_0}$. We call $(D^\bullet, \{\mathfrak{n}_{k_1,k_0}\})$ and $(\widetilde{D}^\bullet, \{\mathfrak{n}_{k_1,k_0}\})$ a filtered A_∞-bimodule over $(C_i^\bullet, \{\mathfrak{m}_k\})$, if

$$\widehat{d}_{\mathfrak{n}} \circ \widehat{d}_{\mathfrak{n}} = 0,$$

where $\widehat{d}_{\mathfrak{n}}$ is the coderivation on $B(C_1[1]^\bullet) \otimes D[1]^\bullet \otimes B(C_0[1]^\bullet)$ determined by $\{\mathfrak{m}_k^{(i)}\}$, $i = 0, 1$, and $\{\mathfrak{n}_{k_1,k_0}\}$. The G-gapped condition is defined in a similar way to the case of filtered A_∞-algebras:

$$\mathfrak{n}_{k_1,k_0} = \sum_{\beta \in G} T^{\mathrm{pr}_1(\beta)} e^{\mathrm{pr}_2(\beta)/2} \mathfrak{n}_{k_1,k_0,\beta},$$

where

$$\mathfrak{n}_{k_1,k_0,\beta} : B_{k_1}(\overline{C}_1[1]^\bullet) \otimes \overline{D}[1]^\bullet \otimes B_{k_0}(\overline{C}_0[1]^\bullet) \to \overline{D}[1]^\bullet.$$

For $\lambda \in \mathbb{R}$, we set

$$\mathcal{F}^\lambda \widetilde{D}^\bullet = T^\lambda \cdot D^\bullet.$$

For $\lambda \geq 0$, $\mathcal{F}^\lambda \widetilde{D}^\bullet \subset D^\bullet$, hence we obtain a filtration on $D^\bullet$. It is clear that

$$\widehat{d}_{\mathfrak{n}}(F^{\lambda_1} B(C_1[1]^\bullet) \otimes \mathcal{F}^{\lambda_2} \widetilde{D}^\bullet \otimes F^{\lambda_3} B(C_0[1]^\bullet)) \subset \mathcal{F}^{\lambda_1+\lambda_2+\lambda_3} \widetilde{D}[1]^\bullet.$$

Note that $\mathfrak{n}_{0,0} \circ \mathfrak{n}_{0,0}$ may not be zero and we have

$$\mathfrak{n}_{0,0} \circ \mathfrak{n}_{0,0}(y) + \mathfrak{n}_{1,0}(\mathfrak{m}_0^{(1)}(1), y) + (-1)^{\deg' y} \mathfrak{n}_{0,1}(y, \mathfrak{m}_0^{(0)}(1)) = 0.$$

For $b_i \in \mathfrak{F}^{\lambda^{(i)}}(C_i[1]^\bullet)$ with $\lambda^{(i)} > 0$, we define

$$\mathfrak{n}_{k_1,k_0}^{b_0,b_1}(\mathbf{x} \otimes y \otimes \mathbf{z}) = \mathfrak{n}_{*,*}(\Phi^{b_1}(\mathbf{x}) \otimes y \otimes \Phi^{b_0}(\mathbf{z})),$$

for $\mathbf{x} \in B_{k_1}(C_1[1]^\bullet)$ and $\mathbf{z} \in B_{k_0}(C_0[1]^\bullet)$. In particular,

$$\mathfrak{n}_{0,0}^{b_0,b_1}(y) = \mathfrak{n}_{*,*}(e^{b_1} \otimes y \otimes e^{b_0}).$$

If b_0, b_1 are solutions of the Maurer-Cartan equations in the filtered A_∞-algebras C_0, C_1, respectively, we find that

$$\mathfrak{n}_{0,0}^{b_0,b_1} \circ \mathfrak{n}_{0,0}^{b_0,b_1} = 0.$$

Namely, we can rectify the original $\mathfrak{n}_{0,0}$ to a coboundary operator $\mathfrak{n}_{0,0}^{b_0,b_1}$ on $D[1]$.

Let $\{\mathfrak{f}_k^{(i)}\}$, $i = 0, 1$, be filtered A_∞-homomorphisms from $C_i^\bullet$ to $C_i^{'\bullet}$ and $(\widetilde{D}^\bullet, \{\mathfrak{n}_{k_1,k_0}\})$, resp. $(\widetilde{D}^{'\bullet}, \{\mathfrak{n}'_{k_1,k_0}\})$, filtered A_∞-bimodules over $(C_i^\bullet, \{\mathfrak{m}_k^{(i)}\})$, resp. $(C_i^{'\bullet}, \{\mathfrak{m}_k{}'(i)\})$. Suppose that there exist a real number c and Λ_{nov}-homomorphisms $\phi_{k_1,k_0} : B_{k_1}(C_1[1]^\bullet) \otimes \widetilde{D}[1]^\bullet \otimes B_{k_0}(C_0[1]^\bullet) \to \widetilde{D}'[1]^\bullet$, $k_1, k_0 = 0, 1, 2, \ldots$, such that

$$\phi_{k_1,k_0}(F^{\lambda_1} B_{k_1}(C_1[1]^\bullet) \otimes \mathcal{F}^{\lambda_2} \widetilde{D}[1]^\bullet \otimes F^{\lambda_3} B_{k_0}(C_0[1]^\bullet)) \subset \mathcal{F}^{\lambda_1+\lambda_2+\lambda_3-c} \widetilde{D}'[1]^\bullet.$$

We call such c the energy loss of $\{\phi_{k_1,k_0}\}$. For such a collection $\{\phi_{k_1,k_0}\}$, we define

$$\widehat{\phi} : B(C_1[1]^\bullet) \otimes \widetilde{D}[1]^\bullet \otimes B(C_0[1]^\bullet) \to B(C'_1[1]^\bullet) \otimes \widetilde{D}'[1]^\bullet \otimes B(C'_0[1]^\bullet)$$

as the homomorphism determined by $\{\mathfrak{f}_k^{(i)}\}$, $i = 0, 1$, and $\{\phi_{k_1,k_0}\}$. We call $\phi = \{\phi_{k_1,k_0}\}$ a weakly filtered A_∞-homomorphism of filtered A_∞-bimodules, if

$$\widehat{d}_{\mathfrak{n}'} \circ \widehat{\phi} = \widehat{\phi} \circ \widehat{d}_{\mathfrak{n}}.$$

When we can take $c = 0$, $\phi = \{\phi_{k_1,k_0}\}$ is called a filtered A_∞-homomorphism. Suppose that $C_i^\bullet, C_i^{'\bullet}$ are G-gapped. Let $G' \subset \mathbb{R} \times 2\mathbb{Z}$ be a G-set such that

$\mathrm{pr}_1|_G^{-1}((-\infty,\lambda])$ is finite for any $\lambda \in \mathbb{R}$ and $\mathrm{pr}_1(G)$ is bounded from below. We say that $\phi = \{\phi_{k_1,k_0}\}$ is G'-gapped, if

$$\phi_{k_1,k_0} = \sum_{\beta' \in G'} T^{\mathrm{pr}_1(\beta')} e^{\mathrm{pr}_2(\beta')/2} \phi_{k_1,k_0,\beta'},$$

where

$$\phi_{k_1,k_0,\beta'} : B_{k_1}(\overline{C}_1[1]^\bullet) \otimes \overline{D}[1]^\bullet \otimes B_{k_0}(\overline{C}_0[1]^\bullet) \to \overline{D}'[1]^\bullet.$$

The homotopy theory between filtered A_∞-homomorphisms of filtered A_∞-bimodules is also developed in [6].

We also proved the Whitehead theorem for (filtered) A_∞-bimodules.

Theorem 2.3 (Theorem 21.35 [6])**.** *Let $\phi : D^\bullet \to D'^\bullet$ be a gapped filtered A_∞-bimodule homomorphism over $(\mathfrak{f}^{(1)}, \mathfrak{f}^{(0)})$, where $\mathfrak{f}^{(i)} : C_i^\bullet \to C_i'^\bullet$ are homotopy equivalences. Suppose that $\phi_{(0,0,\mathbf{0})}$ is a chain homotopy equivalence. Then ϕ is a homotopy equivalence of filtered A_∞-bimodules.*

Remark 2.1. Here we require ϕ is a filtered A_∞-homomorphism. Since a weakly filtered A_∞-homomorphism, $\phi_{(0,0,\mathbf{0})}$ may not induce a chain map with respect to $\mathfrak{n}_{0,0,\mathbf{0}}$ and $\mathfrak{n}'_{0,0,\mathbf{0}}$.

2.6) Filtered $A_{n,K}$ structure.

For a later argument in section 4, we recall the notion of filtered $A_{n,K}$-algebras. Let $G \subset \mathbb{R}_\geq 0 \times 2\mathbb{Z}$ be a monoid as above and $\beta_0 = \mathbf{0} \in G$. For $\beta \in G$, we define

$$\| \beta \| = \begin{cases} \sup\{n | \exists \beta_i \in G \setminus \{\beta_0\},\ \sum_{i=1}^n \beta_i = \beta\} + [\mathrm{pr}_1(\beta)] - 1 & \text{if } \beta \neq \beta_0 \\ -1 & \text{if } \beta = \beta_0, \end{cases}$$

Then we introduce a partial order on $(G \times \mathbb{Z}_{\geq 0}) \setminus \{(\beta_0, 0)\}$ by $(\beta_1, k_1) \succ (\beta_2, k_2)$ if and only if either

$$\| \beta_1 \| + k_1 > \| \beta_2 \| + k_2$$

or

$$\| \beta_1 \| + k_1 = \| \beta_2 \| + k_2, \text{ and } \| \beta_1 \| > \| \beta_2 \| .$$

We write $(\beta_1, k_1) \sim (\beta_2, k_2)$, when

$$\| \beta_1 \| + k_1 = \| \beta_2 \| + k_2, \text{ and } \| \beta_1 \| = \| \beta_2 \| .$$

We define $(\beta_1, k_1) \succsim (\beta_2, k_2)$ if either $(\beta_1, k_1) \succ (\beta_2, k_2)$ or $(\beta_1, k_1) \sim (\beta_2, k_2)$.

We also write $(\beta, k) \prec (n', k')$, when $\| \beta \| + k < n' + k'$ or $\| \beta \| + k = n' + k'$ and $\| \beta \| < n'$.

Let $\overline{C}^\bullet$ be a cochain complex over R and $C^\bullet = \overline{C}^\bullet \otimes \Lambda_{0,nov}$. Suppose that there are

$$\mathfrak{m}_{k,\beta} : B(\overline{C}[1]^\bullet) \to \overline{C}[1]^\bullet$$

for $(\beta, k) \in (G \times \mathbb{Z}) \setminus \{(\beta_0, 0)\}$ with $(\beta, k) \prec (n, K)$. We also suppose that $\mathfrak{m}_{1,\beta_0}$ is the boundary operator of the cochain complex $C^\bullet$. We call $(C^\bullet, \{\mathfrak{m}_{k,\beta}\})$ a G-gapped filtered $A_{n,K}$-algebra, if the following holds

$$\sum_{\beta_1+\beta_2=\beta, k_1+k_2=k+1} \sum_i (-1)^{\deg' \mathbf{x}_i^{(1)}} \mathfrak{m}_{k_2,\beta_2}\big(\mathbf{x}_i^{(1)}, \mathfrak{m}_{k_1,\beta_1}(\mathbf{x}_i^{(2)}), \mathbf{x}_i^{(3)}\big) = 0$$

for all $(\beta, k) \prec (n, K)$, where

$$\Delta^2(\mathbf{x}) = \sum_i \mathbf{x}_i^{(1)} \otimes \mathbf{x}_i^{(2)} \otimes \mathbf{x}_i^{(3)}.$$

Here Δ is the coproduct of the tensor coalgebra.

We also have the notion of filtered $A_{n,K}$-homomorphisms, filtered $A_{n,K}$-homotopy equivalences in a natural way. In [6], we proved the following:

Theorem 2.4 (Theorem 30.72 in [6]). *Let $C_1^\bullet$ be a filtered $A_{n,K}$-algebra and $C_2^\bullet$ a filtered $A_{n',K'}$-algebra such that $(n,K) \prec (n',K')$. Let $\mathfrak{h} : C_1^\bullet \to C_2^\bullet$ be a filtered $A_{n,K}$-homomorphism. Suppose that $\mathfrak{h}$ is a filtered $A_{n,K}$-homotopy equivalence. Then there exist a filtered $A_{n',K'}$-algebra structure on $C_1^\bullet$ extending the given filtered $A_{n,K}$-algebra structure and a filtered $A_{n',K'}$-homotopy equivalence $C_1^\bullet \to C_2^\bullet$ extending the given filtered $A_{n,K}$-homotopy equivalence $\mathfrak{h}$.*

3. Canonical models

In this section, we give the notion of canonical models and explain their construction after section 23, Chapter 5 of [6]. The unfiltered version of such a result goes back to Kadeishvili [10]. There are two methods to construct canonical models in unfiltered case. One is based on obstruction theory due to Kadeishvili and the other uses the summation over trees due to Kontsevich and Soibelman [11]. Our argument is an adaptation of the latter argument and we also constructed the canonical models for filtered case.

When the ground coefficent ring R is a field, we have the following:

Theorem 3.1 (Theorem 23.1, Theorem 23.2 in [6]). *(1) Any unfiltered A_∞-algebra $(\overline{C}^\bullet, \{\overline{\mathfrak{m}}_k\})$ is homotopy equivalent to an A_∞-algebra $(\overline{C'}^\bullet, \{\overline{\mathfrak{m}}'_k\})$ with $\overline{\mathfrak{m}}'_1 = 0$.*
(2) Any gapped filtered A_∞-algebra $(C^\bullet, \{\mathfrak{m}_k\})$ is homotopy equivalent to a gapped filtered A_∞-algebra $(C'^\bullet, \{\mathfrak{m}'_k\})$ with $\overline{\mathfrak{m}}'_1 = 0$. Moreover, the homotopy equivalence can be taken as a gapped A_∞-homomorphism.

An A_∞-algebra is called *canonical*, if $\overline{\mathfrak{m}}_1 = 0$. A canonical model of an A_∞-algebra is a canonical A_∞-algebra homotopy equivalent to the original one. The statement (1) is Kadeishvili's theorem and implies that the $\overline{\mathfrak{m}}_1$-cohomology has a structure of an A_∞-algebra. Note that, in general, we do not have $\mathfrak{m}_1$-cohomology, since $\mathfrak{m}_1 \circ \mathfrak{m}_1$ may not be zero. A filtered A_∞-algebra is called *canonical*, if $\mathfrak{m}_{1,0} = \overline{\mathfrak{m}}_1 = 0$. A canonical model of a filtered A_∞-algebra is a canonical filtered A_∞-algebra homotopy equivalent to the original one.

Pick a submodule $\mathcal{H}^\bullet \overset{\iota}{\hookrightarrow} \ker \overline{\mathfrak{m}}_1 \cap \overline{C}^\bullet$ such that $\iota_* : \mathcal{H}^k \cong \mathrm{H}^k(\overline{C}^\bullet, \overline{\mathfrak{m}}_1)$, and $\Pi^k : \overline{C}^k \to \mathcal{H}^k \subset \overline{C}^k$ such that $\Pi^k \circ \Pi^k = \Pi^k$ and $\Pi^k \circ \overline{\mathfrak{m}}_1 = 0$. We will construct a structure of a filtered A_∞-algebra on $\mathcal{H}[1]^\bullet \otimes \Lambda_{0,nov}$ and a filtered A_∞-homomorphism from $\mathcal{H}[1]^\bullet \otimes \Lambda_{0,nov}$ to $C[1]^\bullet$, which is a weak homotopy equivalence. Since R is a field, any cochain homomorphism inducing an isomorphism on cohomologies is a weak homotopy equivalence. Firstly, we observe the following:

Lemma 3.2. *There exist $G^k : \overline{C}^k \to \overline{C}^{k-1}$, $k = 0, 1, \dots, n$, such that*

$$id - \Pi^k = -(\overline{\mathfrak{m}}_1 \circ G^k + G^{k+1} \circ \overline{\mathfrak{m}}_1), \tag{1}$$

$$G^k \circ G^{k+1} = 0. \tag{2}$$

From now on, let $\mathcal{H}^\bullet \overset{\iota}{\hookrightarrow} \overline{C}^\bullet$ be a subcomplex and $\Pi : \overline{C}^k \to \overline{C}^k$ be a projection to $\mathcal{H}^k$ such that there exist $G^k : \overline{C}^k \to \overline{C}^{k-1}$ satisfying (1), (2) in Lemma 3.2. We do not assume that $\overline{\mathfrak{m}}_1|_{\mathcal{H}} = 0$. Thus $\mathcal{H}^\bullet$ is not necessarily isomorphic to

the cohomology $H^\bullet(\overline{C}^\bullet)$. But the condition (1) implies that $\iota_* : \mathrm{H}^\bullet(\mathcal{H}, \overline{\mathfrak{m}}_1|_{\mathcal{H}}) \cong \mathrm{H}^\bullet(\overline{C}^\bullet, \overline{\mathfrak{m}}_1)$. Theorem 3.1 follows from the following:

Theorem 3.3. *(1) There exists a structure $\{\overline{\mathfrak{m}}'_k\}_{k=1}^\infty$ of an A_∞-algebra on $\mathcal{H}$ with $\mathfrak{m}'_1 = \mathfrak{m}_1|_{\mathcal{H}}$. The inclusion ι extends to an A_∞-homomorphism $\{\overline{\mathfrak{f}}_k\}_{k=1}^\infty$ with $\overline{\mathfrak{f}}_1 = \iota$.*
(2) There exists a structure $\{\mathfrak{m}_k\}_{k=0}^\infty$ of a filtered A_∞-algebra on $\mathcal{H} \otimes \Lambda_{0,nov}$. The inclusion ι extends to a filtered A_∞-homomorphism $\{\mathfrak{f}\}_{k=0}^\infty$ with $\mathfrak{f}_{1,0} = \iota$.

Let G be a monoid as in section 2 and $\mathrm{pr}_1(G) = \{\lambda_{(i)}\}$ such that

$$0 = \lambda_{(0)} < \lambda_{(1)} < \lambda_{(2)} < \cdots \to +\infty,$$

unless $G = \{(0,0)\}$. We write

$$\mathfrak{m}_{k,i} = \sum_{\beta \in G \quad \mathrm{pr}_1(\beta) = \lambda_{(i)}} e^{\mathrm{pr}_2(\beta)/2} \mathfrak{m}_{k,\beta}$$

and

$$\mathfrak{m}^\circ_{k,i} = T^{\lambda_{(i)}} \mathfrak{m}_{k,i}.$$

Thus $\mathfrak{m}_k = \sum_i \mathfrak{m}^\circ_{k,i}$. Here $\mathfrak{m}_{k,i}$ is considered as

$$\mathfrak{m}_{k,i} : B_k(\overline{C}[1]^\bullet) \otimes R[e, e^{-1}] \to \overline{C}[1]^\bullet \otimes R[e, e^{-1}].$$

By an abuse of notation, we also denote by Π^k, G^k the extensions thereof to $\overline{C}^k \otimes R[e, e^{-1}]$ as a $R[e, e^{-1}]$-module homomorphism.

In order to define a G-gapped filtered A_∞-structure on $\mathcal{H}[1]^\bullet \otimes \Lambda_{0,nov}$ and a G-gapped A_∞-homomorphism from $\mathcal{H}[1]^\bullet \otimes \Lambda_{0,nov}$ to $C[1]^\bullet$, we introduce some notation.

A decorated planar rooted tree is a quintet $\Gamma = (T, i, v_0, V_{tad}, \eta)$, which consists of

- T is a tree,
- $i : T \to D^2$ is an embedding,
- v_0 is the root vertex,
- $V_{tad} = \{\text{vertices of valency } 1\} \setminus C^0_{ext}(T)$,
- $\eta : C^0_{int}(T) = C^0(T) \setminus C^0_{ext}(T) \to \{0, 1, 2, \dots\}$.

Here $C^0(T)$ is the set of vertices of the tree T, $C^0_{ext}(T) = i^{-1}(\partial D^2)$ is the set of exterior vertices and $C^0_{int}(T)$ is the set of interior vertices. Note that the root vertex v_0 is an exterior vertex and $V_{tad} \subset C^0_{int}(T)$. Let G^+_k be the set of $\Gamma = (T, i, v_0, V_{tad}, \eta)$ such that $\#C^0_{ext} = k$ and $\eta(v) > 0$ if $v \in C^0_{int}(T)$ is a vertex of valency 1 or 2. We set $E(\Gamma) = \sum_{v \in C^0_{int}(T)} \lambda_{(\eta(v))}$.

For each $\Gamma \in G^+_{k+1}$, we construct

$$\mathfrak{m}_\Gamma : B_k(\mathcal{H}[1]^\bullet) \otimes R[e, e^{-1}] \to \mathcal{H}[1]^\bullet \otimes R[e, e^{-1}],$$

which is of degree 1 and

$$\mathfrak{f}_\Gamma : B_k(\mathcal{H}[1]^\bullet) \otimes R[e, e^{-1}] \to \overline{C}[1]^\bullet \otimes R[e, e^{-1}],$$

which is of degree 0. Then we define

$$\mathfrak{m}'_k = \sum_{\Gamma \in G^+_{k+1}} T^{E(\Gamma)} \mathfrak{m}_\Gamma : B_k(\mathcal{H}[1]^\bullet) \otimes \Lambda_{0,nov} \to \mathcal{H}[1]^\bullet \otimes \Lambda_{0,nov}$$

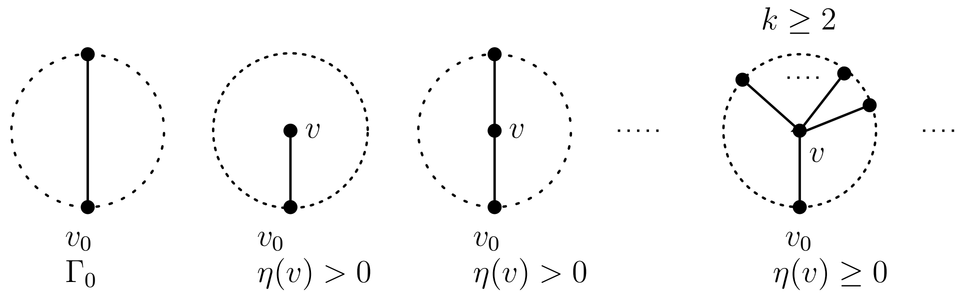

FIGURE 1.

and

$$\mathfrak{f}_k = \sum_{\Gamma \in G^+_{k+1}} T^{E(\Gamma)} \mathfrak{f}_\Gamma : B_k(\mathcal{H}[1]^\bullet) \otimes \Lambda_{0,nov} \to \overline{C}[1]^\bullet \otimes \Lambda_{0,nov}.$$

We will show that $(\mathcal{H}[1]^\bullet \otimes \Lambda_{0,nov}, \{\mathfrak{m}'_k\})$ is a G-gapped filtered A_∞-algebra and $\mathfrak{f} = \{\mathfrak{f}_k\}$ is a G-gapped A_∞-homomorphism, which is a weak homotopy equivalence. Then the Whitehead type theorem implies that $\mathfrak{f}$ is a homotopy equivalence.

Step 1. The case that $\#C^0_{int}(T) = 0$.

Such a T consists of two exterior vertices and an edge joining them. Therefore, there is unique element Γ_0, which belongs to G^+_2. We define

$$\mathfrak{m}_{\Gamma_0} = \overline{\mathfrak{m}}_1|_{\mathcal{H}[1]^\bullet}$$

and

$$\mathfrak{f}_{\Gamma_0} : \mathcal{H}[1]^\bullet \otimes R[e, e^{-1}] \to \overline{C}[1]^\bullet \otimes R[e, e^{-1}]$$

to be the inclusion ι.

Step 2. The case that $\#C^0_{int}(T) = 1$.

For any $k = 0, 1, 2, \dots$, there is a unique planar tree with $\#C^0_{ext}(T) = k+1$ and $\#C^0_{int}(T) = 1$. Let $\Gamma_{k+1} \in G^+_{k+1}$ be a decorated planar tree with one interior vertex v, see Figure 1.

We define

$$\mathfrak{m}_{\Gamma_{k+1}} = \Pi \circ \mathfrak{m}_{k,\eta(v)} : B_k(\mathcal{H}[1]^\bullet) \otimes R[e, e^{-1}] \to \mathcal{H}[1]^\bullet \otimes R[e, e^{-1}]$$

and

$$\mathfrak{f}_{\Gamma_{k+1}} = G \circ \mathfrak{m}_{k,\eta(v)} : B_k(\mathcal{H}[1]^\bullet) \otimes R[e, e^{-1}] \to \overline{C}[1]^\bullet \otimes R[e, e^{-1}].$$

Since the degree of Π, resp. G, is 0, resp. -1, $\mathfrak{m}_{\Gamma_{k+1}}$, resp. $\mathfrak{f}_{\Gamma_{k+1}}$, is of degree 1, resp. 0.

Step 3. General case.

Let v_1 is the vertex closest to the root vertex v_0. Cut the decorated planar tree at v_1, then Γ is decomposed into decorated planar trees $\Gamma^{(1)}, \dots, \Gamma^{(\ell)}$ and an interval toward v_0 in counter-clockwise order, see Figure 2.

Then we define

$$\mathfrak{m}_\Gamma = \Pi \circ \mathfrak{m}_{\ell,\eta(v_1)} \circ (\mathfrak{f}_{\Gamma^{(1)}} \otimes \cdots \otimes \mathfrak{f}_{\Gamma^{(\ell)}})$$

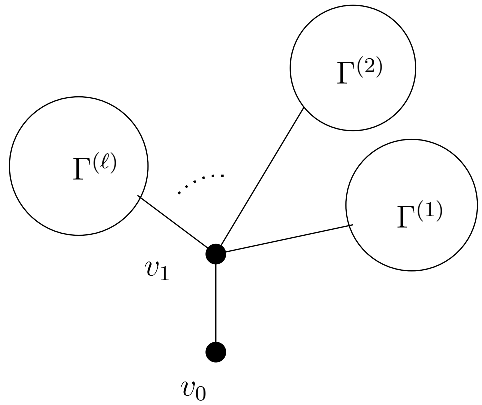

FIGURE 2.

and

$$\mathfrak{f}_\Gamma = G \circ \mathfrak{m}_{\ell,\eta(v_1)} \circ (\mathfrak{f}_{\Gamma^{(1)}} \otimes \cdots \otimes \mathfrak{f}_{\Gamma^{(\ell)}}).$$

Finally we define

$$\mathfrak{m}'_k = \sum_{\Gamma \in G^+_{k+1}} T^{E(\Gamma)} \mathfrak{m}_\Gamma$$

and

$$\mathfrak{f}_k = \sum_{\Gamma \in G^+_{k+1}} T^{E(\Gamma)} \mathfrak{f}_\Gamma.$$

As in §2.1, we obtain a graded coderivation

$$\widehat{d'} = \sum_k \widehat{\mathfrak{m}}'_k : B(\mathcal{H}[1]^\bullet) \otimes \Lambda_{0,nov} \to B(\mathcal{H}[1]^\bullet) \otimes \Lambda_{0,nov}$$

and a (formal) coalgebra homomorphism

$$\widehat{\mathfrak{f}} : B(\mathcal{H}[1]^\bullet) \otimes \Lambda_{0,nov} \to B(C[1]^\bullet).$$

We will show the following:

Proposition 3.4.

$$\widehat{\mathfrak{f}} \circ \widehat{d'} = \widehat{d} \circ \widehat{\mathfrak{f}},$$

where $\widehat{d} = \sum_k \widehat{\mathfrak{m}}_k : B(C[1]^\bullet) \to B(C[1]^\bullet)$.

Since $\overline{\mathfrak{f}}_1 = \mathfrak{f}_{\Gamma_0}$ is the inclusion, we find that $\widehat{\mathfrak{f}}$ is injective using the energy filtration and the number filtration on the bar complex. Then $\widehat{d'} \circ \widehat{d'} = 0$ follows from $\widehat{d} \circ \widehat{d} = 0$. Hence we obtain the following:

Corollary 3.5. *(1)* $(\mathcal{H}[1]^\bullet \otimes \Lambda_{0,nov}, \{\mathfrak{m}'_k\})$ *is a G-gapped filtered A_∞-algebra.*
(2) $\widehat{\mathfrak{f}}$ *is a G-gapped A_∞-homomorphism from* $(\mathcal{H}[1]^\bullet \otimes \Lambda_{0,nov}, \{\mathfrak{m}'_k\})$ *to* $(C[1]^\bullet, \{\mathfrak{m}_k\})$.

The rest of this section is devoted to the proof of Proposition 3.4, which is equivalent to that

$$\mathfrak{f}\circ\widehat{d'} = \mathfrak{m}\circ\widehat{\mathfrak{f}}$$

as maps $B(\mathcal{H}[1]^\bullet)\otimes\Lambda_{0,nov}\to C[1]^\bullet$, where

$$\mathfrak{f} : B(\mathcal{H}[1]^\bullet)\otimes\Lambda_{0,nov}\xrightarrow{\widehat{\mathfrak{f}}} B(C[1]^\bullet)\xrightarrow{\mathrm{pr}} C[1]^\bullet,$$

and

$$\mathfrak{m} : B(C[1]^\bullet)\xrightarrow{\widehat{d}} B(C[1]^\bullet)\xrightarrow{\mathrm{pr}} C[1]^\bullet.$$

Namely, $\mathfrak{f}|_{B_k(\mathcal{H}[1]^\bullet)\otimes\Lambda_{0,nov}} = \mathfrak{f}_k$, $\mathfrak{m}|_{B_k(\mathcal{H}[1]^\bullet)\otimes\Lambda_{0,nov}} = \mathfrak{m}_k$.

We introduce an order on $\{(k,i)|k,i=0,1,2,\dots\}$ by $(k_1,i_1)\prec(k_2,i_2)$ if either $i_1<i_2$ or $i_1=i_2$ and $k_1<k_2$. We show the following claim by the induction on (k,i).

Claim (k,i) .

$$\mathfrak{f}\circ\widehat{d'}\equiv\mathfrak{m}\circ\widehat{\mathfrak{f}} \mod T^{\lambda_{(i+1)}}\cdot C[1]^\bullet \text{ on } B_k(\mathcal{H}[1]^\bullet)\otimes\Lambda_{0,nov}.$$

The key ingredients in the proof are the following relations presented in Figures 3, 4.

Firstly, we consider the case that $i=0$. Claim $(0,0)$ follows from the gapped condition. By the choice of $\mathcal{H}$, Claim (1,0) holds clearly. Suppose that Claim $(\ell,0)$ holds for $\ell<k$. Note that

$$\overline{\mathfrak{f}}_k = \mathfrak{f}^\circ_{k,0} = \Big(\sum_{1<\ell\le k} G\circ\overline{\mathfrak{m}}_\ell\circ\widehat{\overline{\mathfrak{f}}} + \delta_{k1}\mathfrak{f}_{\Gamma_0}\Big)|_{B_k(\mathcal{H}[1]^\bullet)}$$

and

$$\overline{\mathfrak{m}}'_k = \mathfrak{m}^\circ_{k,0} = \Big(\sum_{1<\ell\le k}\Pi\circ\overline{\mathfrak{m}}_\ell\circ\widehat{\overline{\mathfrak{f}}} + \delta_{k1}\mathfrak{m}_{\Gamma_0}\Big)|_{B_k(\mathcal{H}[1]^\bullet)}.$$

Here δ_{ij} is Kronecker's delta. Recall that $\mathfrak{m}_{\Gamma_0}=\mathfrak{m}_{1,0}|_{\mathcal{H}}$ and $\mathfrak{f}_{\Gamma_0}$ is the inclusion. Note also that the restriction of $\widehat{\overline{\mathfrak{f}}}$ to $B_k(\mathcal{H}[1]^\bullet)$ in the right hand sides is determined by $\overline{\mathfrak{f}}_1,\dots,\overline{\mathfrak{f}}_{k-1}$.

Thus we have

$$\begin{aligned}
\overline{\mathfrak{m}}\circ\widehat{\overline{\mathfrak{f}}}|_{B_k(\mathcal{H}[1]^\bullet)} &= \Big(\overline{\mathfrak{m}}_1\circ\widehat{\overline{\mathfrak{f}}} + \sum_{1<\ell\le k}\overline{\mathfrak{m}}_\ell\circ\widehat{\overline{\mathfrak{f}}}\Big)|_{B_k(\mathcal{H}[1]^\bullet)}\\
&= \Big(\sum_{1<\ell\le k}\overline{\mathfrak{m}}_1\circ G\circ\overline{\mathfrak{m}}_\ell\circ\widehat{\overline{\mathfrak{f}}} + \delta_{k1}\overline{\mathfrak{m}}_1\circ\mathfrak{f}_{\Gamma_0} + \sum_{1<\ell\le k}\overline{\mathfrak{m}}_\ell\circ\widehat{\overline{\mathfrak{f}}}\Big)|_{B_k(\mathcal{H}[1]^\bullet)}.\\
&= \Big(\sum_{1<\ell\le k}\Pi\circ\overline{\mathfrak{m}}_\ell\circ\widehat{\overline{\mathfrak{f}}} - \sum_{1<\ell\le k}G\circ\overline{\mathfrak{m}}_1\circ\overline{\mathfrak{m}}_\ell\circ\widehat{\overline{\mathfrak{f}}} + \delta_{k1}\overline{\mathfrak{m}}_1\circ\mathfrak{f}_{\Gamma_0}\Big)|_{B_k(\mathcal{H}[1]^\bullet)}\\
&= \Big(\mathfrak{f}_{\Gamma_0}\circ\overline{\mathfrak{m}}'_k - \sum_{1<\ell\le k}G\circ\overline{\mathfrak{m}}_1\circ\overline{\mathfrak{m}}_\ell\circ\widehat{\overline{\mathfrak{f}}}\Big)|_{B_k(\mathcal{H}[1]^\bullet)}\\
&= \Big(\mathfrak{f}_{\Gamma_0}\circ\overline{\mathfrak{m}}'_k + \sum_{1<\ell'\le k}G\circ\overline{\mathfrak{m}}_{\ell'}\circ\widehat{\overline{d}}\circ\widehat{\overline{\mathfrak{f}}}\Big)|_{B_k(\mathcal{H}[1]^\bullet)}.
\end{aligned}$$

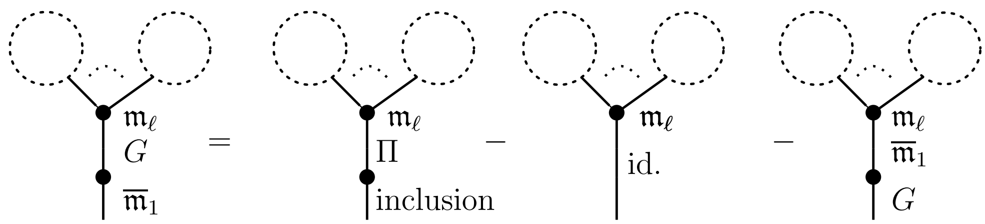

FIGURE 3.

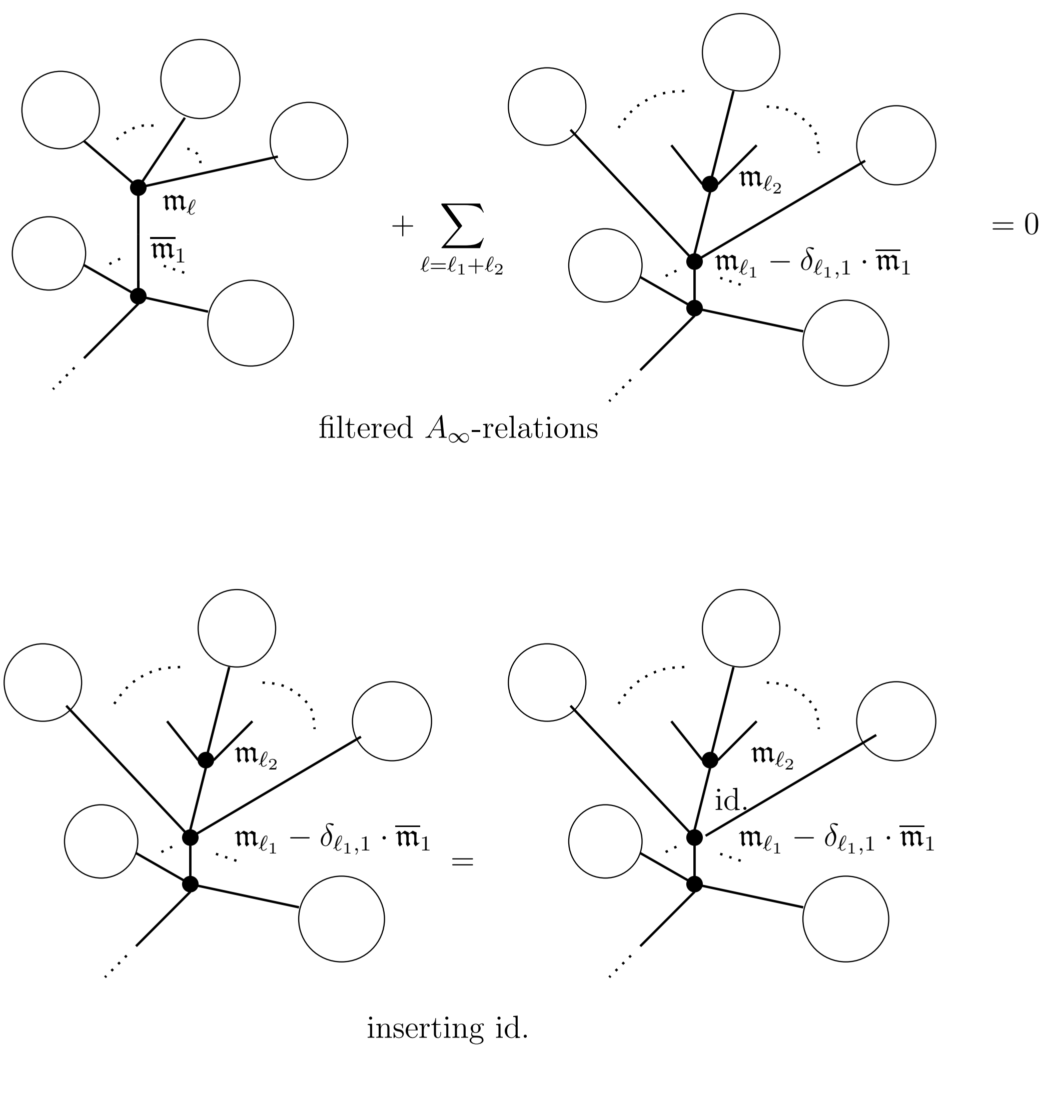

FIGURE 4.

Here we used the fact that $\overline{\mathfrak{m}}_1\circ G+G\circ\overline{\mathfrak{m}}_1=\Pi-id$ and the A_∞-relation $\widehat{\overline{d}}\circ\widehat{\overline{d}}=0$. Since we assumed Claim $(\ell,0)$ for $\ell<k$, i.e.,

$$\overline{\mathfrak{m}}\circ\widehat{\overline{\mathfrak{f}}}=\overline{\mathfrak{f}}\circ\widehat{\overline{d}}' \text{ on } B_\ell(\mathcal{H}[1]^\bullet),$$

we have

$$\widehat{\overline{d}}\circ\widehat{\overline{\mathfrak{f}}}\equiv\widehat{\overline{\mathfrak{f}}}\circ\widehat{\overline{d}}' \mod \overline{C}[1]^\bullet=B_1(\overline{C}[1]^\bullet) \text{ on } B_k(\mathcal{H}[1]^\bullet).$$

Therefore we find that

$$\Big(\sum_{1<\ell'\le k}G\circ\overline{\mathfrak{m}}_{\ell'}\circ\widehat{\overline{d}}\circ\widehat{\overline{\mathfrak{f}}}\Big)|_{B_k(\mathcal{H}[1]^\bullet)}=\Big(\sum_{1<\ell'\le k}G\circ\overline{\mathfrak{m}}_{\ell'}\circ\widehat{\overline{\mathfrak{f}}}\circ\widehat{\overline{d}}'\Big)|_{B_k(\mathcal{H}[1]^\bullet)}.$$

Hence we showed Claim $(k,0)$, i.e.,

$$\overline{\mathfrak{m}}\circ\widehat{\overline{\mathfrak{f}}}=\overline{\mathfrak{f}}\circ\widehat{\overline{d}}'$$

on $B_k(\mathcal{H}[1]^\bullet)$.

Next we assume that Claim (k,i) holds for all $k=0,1,2,\ldots$. We prove Claim $(k,i+1)$ by the induction on k. Note that Case 3-1 below does not occur in the case that $k=0$.

First of all, we recall from the definition of G^+_{k+1} that

$$\mathfrak{f}_k=\sum_{\Gamma\in G^+_{k+1}}T^{E(\Gamma)}\mathfrak{f}_\Gamma=\sum_{(\ell,j)\ne(1,0)}G\circ\mathfrak{m}^\circ_{\ell,j}\circ\widehat{\mathfrak{f}}|_{B_k(\mathcal{H}[1]^\bullet)\otimes\Lambda_{0,nov}}+\delta_{k1}\mathfrak{f}_{\Gamma_0}.$$

Then we have

$$\begin{aligned}
\mathfrak{m}\circ\widehat{\mathfrak{f}}|_{B_k(\mathcal{H}[1]^\bullet)}=&\Big(\mathfrak{m}_{1,0}\circ\widehat{\mathfrak{f}}+\sum_{(\ell,j)\ne(1,0)}\mathfrak{m}^\circ_{\ell,j}\circ\widehat{\mathfrak{f}}\Big)|_{B_k(\mathcal{H}[1]^\bullet)}\\
=&\Big(\sum_{(\ell,j)\ne(1,0)}\mathfrak{m}_{1,0}\circ G\circ\mathfrak{m}^\circ_{\ell,j}\circ\widehat{\mathfrak{f}}+\delta_{k1}\mathfrak{m}_{1,0}\circ\mathfrak{f}_{\Gamma_0}\\
&+\sum_{(\ell,j)\ne(1,0)}\mathfrak{m}^\circ_{\ell,j}\circ\widehat{\mathfrak{f}}\Big)|_{B_k(\mathcal{H}[1]^\bullet)}.\\
=&\Big(\sum_{(\ell,j)\ne(1,0)}\Pi\circ\mathfrak{m}^\circ_{\ell,j}\circ\widehat{\mathfrak{f}}-\sum_{(\ell,j)\ne(1,0)}G\circ\mathfrak{m}_{1,0}\circ\mathfrak{m}^\circ_{\ell,j}\circ\widehat{\mathfrak{f}}\\
&+\delta_{k1}\mathfrak{m}_{1,0}\circ\mathfrak{f}_{\Gamma_0}\Big)|_{B_k(\mathcal{H}[1]^\bullet)}\\
=&\Big(\mathfrak{f}_{\Gamma_0}\circ\mathfrak{m}'_k-\sum_{(\ell,j)\ne(1,0)}G\circ\mathfrak{m}_{1,0}\circ\mathfrak{m}^\circ_{\ell,j}\circ\widehat{\mathfrak{f}}\Big)|_{B_k(\mathcal{H}[1]^\bullet)}\\
=&\Big(\mathfrak{f}_{\Gamma_0}\circ\mathfrak{m}'_k+\sum_{(\ell',j')\ne(1,0)}G\circ\mathfrak{m}^\circ_{\ell',j'}\circ\widehat{d}\circ\widehat{\mathfrak{f}}\Big)|_{B_k(\mathcal{H}[1]^\bullet)}.
\end{aligned}$$

In the third equality, we used the fact that $\mathfrak{m}_{1,0}\circ G+G\circ\mathfrak{m}_{1,0}=\Pi-id$.

We will show that

$$\sum_{(\ell',j')\ne(1,0)}G\circ\mathfrak{m}^\circ_{\ell',j'}\circ\widehat{d}\circ\widehat{\mathfrak{f}}\equiv\sum_{(\ell',j')\ne(1,0)}G\circ\mathfrak{m}^\circ_{\ell',j'}\circ\widehat{\mathfrak{f}}\circ\widehat{d}' \mod T^{\lambda_{(i+2)}},$$

which implies that

$$\mathfrak{m}\circ\widehat{\mathfrak{f}}\equiv\widehat{\mathfrak{f}}\circ\widehat{d}' \mod T^{\lambda_{(i+2)}}.$$

Case 1: $\ell' = 0$**.** Note that the $B_0(C[1]^\bullet) = \Lambda_{0,nov}$-components of Im $\widehat{d} \circ \widehat{\mathfrak{f}}$ and Im $\widehat{\mathfrak{f}} \circ \widehat{d'}$ are zero. Hence we have

$$\mathfrak{m}^\circ_{0,j'} \circ \widehat{d} \circ \widehat{\mathfrak{f}} = \mathfrak{m}^\circ_{0,j'} \circ \widehat{\mathfrak{f}} \circ \widehat{d'} = 0.$$

Case 2: $\ell' = 1$**.** For $j' \neq 0$, $\mathfrak{m}^\circ_{1,j'} \equiv 0 \mod T^{\lambda_{(1)}}$. By the induction hypothesis, we have

$$\widehat{\mathfrak{f}} \circ \widehat{d'} \equiv \widehat{d} \circ \widehat{\mathfrak{f}} \mod T^{\lambda_{i+1}}.$$

Since $\lambda_{(i+2)} \leq \lambda_{(i+1)} + \lambda_{(1)}$, we obtain

$$\mathfrak{m}^\circ_{1,j'} \circ \widehat{\mathfrak{f}} \circ \widehat{d'} \equiv \mathfrak{m}^\circ_{1,j'} \circ \widehat{d} \circ \widehat{\mathfrak{f}} \mod T^{\lambda_{(i+2)}}.$$

Case 3: $\ell' \geq 2$**.** Let $\mathbf{x} \in B_k(\mathcal{H}[1]^\bullet) \otimes \Lambda_{0,nov}$. Write

$$\Delta^{\ell'-1}\mathbf{x} = \sum_a \mathbf{x}_{1,a} \otimes \cdots \otimes \mathbf{x}_{\ell',a},$$

where Δ is the coproduct and $\mathbf{x}_{i,a} \in B_{k_{i,a}}(\mathcal{H}[1]^\bullet) \otimes \Lambda_{0,nov}$. Then we have

$$\begin{aligned}\widehat{\mathfrak{f}} \circ \widehat{d'}(\mathbf{x}) = \sum_a \sum_j (-1)^{\deg' \mathbf{x}_{1,a} + \cdots + \deg' \mathbf{x}_{j-1,a}} \mathfrak{f}_{k_{1,a}}(\mathbf{x}_{1,a}) \otimes \cdots \otimes \mathfrak{f}_{k_{j,a}}(\widehat{d'}(\mathbf{x}_{j,a})) \otimes \cdots \\ \cdots \otimes \mathfrak{f}_{k_{\ell',a}}(\mathbf{x}_{\ell',a}).\end{aligned}$$

Case 3-1: $k_{j,a} < k$**.** In this case, we have

$$\mathfrak{f}_{k_{j,a}}(\widehat{d'}(\mathbf{x}_{j,a}) \equiv \mathfrak{m} \circ \widehat{\mathfrak{f}}(\mathbf{x}_{j,a}) \mod T^{\lambda_{(i+2)}}$$

by the induction hypothesis. Hence

$$\begin{aligned}&\mathfrak{f}_{k_{1,a}}(\mathbf{x}_{1,a}) \otimes \cdots \otimes \mathfrak{f}_{k_{j,a}}(\widehat{d'}(\mathbf{x}_{j,a})) \otimes \cdots \otimes \mathfrak{f}_{k_{\ell',a}}(\mathbf{x}_{\ell',a}) \\ \equiv &\mathfrak{f}_{k_{1,a}}(\mathbf{x}_{1,a}) \otimes \cdots \otimes \mathfrak{m} \circ \widehat{\mathfrak{f}}(\mathbf{x}_{j,a}) \otimes \cdots \otimes \mathfrak{f}_{k_{\ell',a}}(\mathbf{x}_{\ell',a}) \mod T^{\lambda_{(i+2)}}.\end{aligned}$$

Case 3-2: $k_{j,a} = k$**.** In this case, $k_{j',a} = 0$ for $j' \neq j$, i.e., $\mathbf{x}_{j',a} \in B_0(\mathcal{H}[1]^\bullet) \otimes \Lambda_{0,nov}$. Without loss of generality, we may assume that $\mathbf{x}_{j',a} = 1$ for $j' \neq j$.

By the induction hypothesis, we have

$$\mathfrak{f}(\widehat{d'}(\mathbf{x}_{j,a})) \equiv \mathfrak{m}(\widehat{\mathfrak{f}}(\mathbf{x}_{j,a})) \mod T^{\lambda_{(i+1)}},$$

which implies that

$$\mathfrak{f}_0(1) \otimes \cdot \otimes \mathfrak{f}(\widehat{d'}(\mathbf{x}_{j,a})) \otimes \cdots \otimes \mathfrak{f}_1(1) \equiv \mathfrak{f}_0(1) \otimes \cdot \otimes \mathfrak{m}(\widehat{\mathfrak{f}}(\mathbf{x}_{j,a})) \otimes \cdots \otimes \mathfrak{f}_1(1) \mod T^{\lambda_{(i+2)}}.$$

Here we used $\mathfrak{f}_0(1) \equiv 0 \mod T^{\lambda_{(1)}}$.

In sum, we obtain Claim $(k, i+1)$ for all k.

By the construction, $\overline{\mathfrak{f}}_1$ is a chain homotopy equivalence (Π is a homotopy inverse). Therefore, Theorem 2.2 implies that $\{\mathfrak{f}_k\}$ is a homotopy equivalence of filtered A_∞-algebras.

4. Filtered A_∞-algebra associated to Lagrangian submanifolds

Let (M,ω) be a closed symplectic manifold and L a Lagrangian submanifold. We only consider the case that L is an embedded compact Lagrangian submanifold without boundary equipped with a relative spin structure, see §44 in [6]. We constructed a filtered A_∞-algebra associated to L in (M,ω). As we explained in section 2, the framework of (filtered) A_∞-algebras, bimodules, etc. is adequate to formulate the condition under which Floer complex is obtained.

In this section, we briefly recall the way of constructing filtered A_∞-algebra associated to L. Although the readers may find Proposition 4.1 below too technical, we present it precisely so that we can explain how to modify it for the purpose of section 5.

A naive idea of the construction is to use the moduli space of pseudo-holomorphic discs to *deform* the intersection products of chains in L in a similar way to the quantum cohomology, where the intersection product on (co)homology is deformed by the moduli space of pseudo-holomorphic spheres, more precisely, stable maps of genus 0. Here appears a difference: while the moduli spaces of stable maps of genus 0 are (virtual) cycles, the moduli spaces of stable bordered stable maps are, in general, not (virtual) cycles, but with codimension 1 boundary (in the sense of Kuranishi structure). Therefore, we cannot restrict ourselves to cycles and forced to work with chains. However, the intersection product is not defined in chain level, e.g., the self intersection of chains. We start with a subcomplex of the singular chain complex such that the inclusion induces an isomorphism on homology. Then take *perturbed* intersection product of generators of the subcomplex and add them to get a larger subcomplex such that the inclusion induces an isomorphism on homology. Once we get such nested subcomplexes, we apply the argument in the proof of Theorem 3.3 to define the operation $\overline{\mathfrak{m}}_2$ on a fixed subcomplex. This multiplicative structure is not associative, but associative up to homotopy. So we proceed to constructed other operations $\overline{\mathfrak{m}}_k$ in a similar way, see Corollary 30.89 in section 30.6, [6] for a detailed argument. In this way, we obtain an A_∞-algebra.

For the construction of the filtered A_∞-algebra, we include the effect from the moduli space of bordered stable maps. We need to take perturbations of the moduli spaces to define the operations not only perturbation in the intersection product mentioned above. Our strategy is to construct an $A_{n,K}$-algebra on $C_{(g)}(L)$, which is generated by $\chi_{(g)}$ in Proposition 4.1, for a sufficiently large g. Then we use the obstruction theory to extend a filtered $A_{n,K}$-structure to a filtered $A_{n',K'}$-structure $((n,K) \precsim (n',K'))$. The resulting filtered A_∞-structure is unique up to homotopy, see §30 in Chapter 7, [6].

Let $\mu_L \in \mathrm{H}^2(M,L;\mathbb{Z})$ be the Maslov class of the Lagrangian submanifold L. We introduce an equivalence relation $\sim$ on $\mathrm{H}_2(M,L;\mathbb{Z})$ by $\beta_1 \sim \beta_2$ if and only if $\omega(\beta_1) = \omega(\beta_2)$ and $\mu_L(\beta_1) = \mu_L(\beta_2)$.

Pick an almost complex structure J compatible with ω. Denote by $\mathcal{M}(\beta; L, J)$ the moduli space of bordered stable maps $u : (\Sigma, \partial\Sigma) \to (M, L)$ of genus 0 representing β and by $\mathcal{M}_{k+1}(\beta; L, J)$ be the moduli space of bordered stable maps in the class β of genus 0 with $k+1$ marked points $z_0, z_1, \dots, z_k$ on the regular part of $\partial\Sigma$. Denote by $\mathcal{M}^{\mathrm{main}}_{k+1}(\beta; L, J)$ the component, on which the marked points $z_0, z_1, \dots, z_k$ respect the *counter-clockwise* cyclic order on the boundary of bordered semi-stable curve of genus 0 with connected boundary. Let $\mathfrak{G}(L)$ be the monoid contained in $\Pi(M,L)$ generated by β with $\mathcal{M}(\beta; L, J) \neq \emptyset$. We write $\beta_0 = 0 \in \mathfrak{G}(L)$.

Our basic idea is as follows. For singular simplices $P_1, \dots, P_k$ in L, we consider the fiber product in the sense of Kuranishi structure

$$\mathcal{M}^{\mathrm{main}}_{k+1}(\beta; P_1, \dots, P_k) = \mathcal{M}^{\mathrm{main}}_{k+1}(\beta; L, J)_{\mathbf{ev}} \times_{L^k} (P_1 \times \cdots \times P_k),$$

where $\mathbf{ev} = (ev_1, \dots, ev_k)$ is the evaluation map at $z_1, \dots, z_k$. (For the orientation issue, see Chapter 9 [6].) Then we would like to define

$$\mathfrak{m}_{k,\beta}(P_1, \dots, P_k) = (ev_0 : \mathcal{M}^{\mathrm{main}}_{k+1}(\beta; P_1, \dots, P_k) \to L),$$

where ev_0 is the evaluation at z_0.

Note that $\mathcal{M}^{\mathrm{main}}_{k+1}(\beta)$ is not necessarily a manifold or an orbifold and that ev_i are not necessarily submersions even if $\mathcal{M}^{\mathrm{main}}_{k+1}(\beta)$ is such a nice space. In order to deal with this issue, we introduced the notion of Kuranishi structure [8], see also Appendix in [6]. Here is a digression on Kuranishi structure.

Let X be a compact Hausdorff space. A Kuranishi structure on X consists of a covering of X by Kuranishi neighborhoods of the same *virtual* dimension and coordinate changes among them. A Kuranishi neighborhood around $p \in X$ is a quintet $(V_p, E_p, \Gamma_p, s_p, \psi_p)$, where

- V_p is a smooth manifold of finite dimension,
- E_p is a real vector bundle over V_p of finite rank,
- Γ_p is a finite group acting smoothly and effectively on V_p and E_p such that $E_p \to V_p$ is a Γ_p-equivariant vector bundle,
- s_p is a Γ_p-equivariant section of $E_p \to V_p$,
- ψ_p is a homeomorphism from $s_p^{-1}(0)/\Gamma_p$ to a neighborhood of p in X.

The vector bundle $E_p \to V_p$ is called the *obstruction* bundle and the section s_p the Kuranishi map. We have coordinate changes among Kuranishi neighborhoods, see [8], [6]. We require that $\dim V_p - \operatorname{rank} E_p$ does not depend on $p \in X$ and call it the *virtual* dimension of the space X equipped with Kuranishi structure.

The moduli spaces of stable maps, bordered stable maps carry Kuranishi structures, hence we can locally describe the moduli space as $s_p^{-1}(0)/\Gamma_p$ in the definition of Kuranishi neighborhoods. If s_p is transversal to the zero section, the moduli space is locally an orbifold. In general, we cannot perturb s_p to a Γ_p-equivariant section s_p', which is transversal to the zero section. Instead of single valued sections, we consider perturbations by Γ_p-equivariant *multi-valued* sections, each branch of which is transversal to the zero section. Then we arrange them compatible under the coordinate change. In this way, we obtain *perturbed* moduli spaces.

We take a multi-valued perturbation $\mathfrak{s}$ of Kuranishi maps for $\mathcal{M}^{\mathrm{main}}_{k+1}(\beta; P_1, \dots, P_k)$ such that each branch of $\mathfrak{s}$ is transversal to the zero section. After taking a triangulation of the perturbed zero locus $\mathcal{M}^{\mathrm{main}}_{k+1}(\beta; P_1, \dots, P_k)^{\mathfrak{s}}$ of $\mathfrak{s}$, we obtain a *virtual* chain

$$ev_0 : \mathcal{M}^{\mathrm{main}}_{k+1}(\beta; P_1, \dots, P_k)^{\mathfrak{s}} \to L.$$

To make this argument rigorous, we build a sequence of subcomplexes of the singular chain complex of L and a series of operations $\mathfrak{m}^{geo}_{k,\beta}$. For details, see Chapter 7 in [6]. Here we briefly recall a part of it, in particular, the construction of a series of subcomplexes of singular chain complex of L. In section 5, we explain how to arrange this construction in relation with the Morse theory.

In §30 in [6], we constructed countable sets $\chi_g(L)$ of singular C^∞-simplices on L. For a simplex $P \in \chi_g(L)$, we call g the generation of P. Write

$$\chi_{(g)} = \bigcup_{g' \le g} \chi_{g'}(L)$$

and denote by $C_{(g)}(L;R)$ the R-vector space generated by $\chi_{(g)}(L)$. Let $S(L;R)$ be the singular C^∞-chain complex of L with coefficients in R.

Condition 1. Any face of $P \in \chi_g(L)$ belongs to $\chi_{(g)}(L)$.

Condition 2. The inclusion $C_{(g)}(L) \to S(L;R)$ induces an isomorphism on homology.

For $\beta \in \mathfrak{G}(L)$, we define

$$\| \beta \| = \begin{cases} \sup\{n | \exists \beta_1, \dots, \beta_n \in \mathfrak{G}(L) \setminus \{\beta_0\}, \sum_{i=1}^n \beta_i = \beta\} + [\omega(\beta)] - 1 & \text{if } \beta \ne \beta_0 \\ -1 & \text{if } \beta = \beta_0 \end{cases}$$

Here $[\omega(\beta)]$ is the largest integer not greater than $\omega(\beta)$.

By Gromov's compactness, the number of $\beta \in \mathfrak{G}(L)$ with $\| \beta \| \le C$ is finite for any C.

Next we introduce an additional data $\mathfrak{d} : \{1, \dots, k\} \to \mathbb{Z}_{\ge 0}$, which is called a decoration. For a pair $(\mathfrak{d}, \beta)$ such that $\mathcal{M}^{\text{main}}_{k+1}(\beta) \ne \emptyset$, we define

$$\| (\mathfrak{d}, \beta) \| = \begin{cases} \max_{i \in \{1,\dots,k\}} \mathfrak{d}(i) + \| \beta \| + k & \text{if } k \ne 0 \\ \| \beta \| & \text{if } k = 0. \end{cases}$$

We will take the fiber product of $\mathcal{M}^{\text{main}}_{k+1}(\beta)$ and singular simplices P_i in L. The decoration $\mathfrak{d}$ is introduced in order to include the generations of singular simplices P_i into the data. When we emphasize that the decoration $\mathfrak{d}$ is equipped with the moduli space $\mathcal{M}^{\text{main}}_{k+1}(\beta)$, we denote it by $\mathcal{M}^{\text{main},\mathfrak{d}}_{k+1}(\beta)$.

Proposition 4.1 (Proposition 30.35 in [6])**.** *For any $\delta > 0$ and $\mathcal{K} > 0$, there exist $\chi_{(g)}(L)$, $g = 0, \dots, \mathcal{K}$, and multisections $\mathfrak{s}_{\mathfrak{d},k,\beta,\vec{P}}$ for $\| (\mathfrak{d}, \beta) \| \le \mathcal{K}$ with the following properties:*

- *$\chi_{(g)}(L)$ satisfies Conditions 1 and 2 above.*
- *Let $P_i \in \chi_{\mathfrak{d}(i)}(L), i = 1, \dots, k$. We put*

$$\mathcal{M}^{\text{main},\mathfrak{d}}_{k+1}(\beta; P_1, \dots, P_k) = \mathcal{M}^{\text{main},\mathfrak{d}}_{k+1}(\beta) \times_{L^k} \prod P_i$$

  and define a multisection $\mathfrak{s}_{\mathfrak{d},k,\beta,\vec{P}}$ thereof. $\mathfrak{s}_{\mathfrak{d},k,\beta,\vec{P}}$ is transversal to the zero section.
- *If $g = \| (\mathfrak{d}, \beta) \|$, then*

$$ev_{0*}\big(\mathcal{M}^{\text{main},\mathfrak{d}}_{k+1}(\beta; P_1, \dots, P_k)^{\mathfrak{s}_{\mathfrak{d},k,\beta,\vec{P}}}\big)$$

  is decomposed into elements of $\chi_{(g)}(L)$. Here and henceforth we denote

$$\mathcal{M}^{\text{main},\mathfrak{d}}_{k+1}(\beta; P_1, \dots, P_k)^{\mathfrak{s}_{\mathfrak{d},k,\beta,\vec{P}}} := \mathfrak{s}^{-1}_{\mathfrak{d},k,\beta,\vec{P}}(0).$$

- *The multisections $\mathfrak{s}_{\mathfrak{d},k,\beta,\vec{P}}$ satisfy certain compatibility conditions.*
- *The zero locus $\mathfrak{s}^{-1}_{\mathfrak{d},k,\beta,\vec{P}}(0)$ is in a δ-neighborhood of the zero locus of the original Kuranishi map.*

For the compatibility conditions in the above statement, see Conditions 30.38 and 30.44 in [6].

Now we explain the way of constructing the filtered A_∞-algebra associated to L. We put

$$\mathfrak{m}^{geo}_{k,\beta}(P_1,\dots,P_k) = (ev_0 : \mathcal{M}^{\mathrm{main},\mathfrak{d}}_{k+1}(\beta; P_1,\dots,P_k)^{\mathfrak{s}_{\mathfrak{d},k,\beta,\vec{P}}} \to L),$$

when $P_i \in \chi_{(\mathfrak{d}(i))}$, $i = 1,\dots,k$. Then $\mathfrak{m}^{geo}_{k,\beta}(P_1,\dots,P_k)$ is decomposed into elements of $\chi_{(g)}$, where $g = \| (\mathfrak{d},\beta) \|$. Using the idea in section 3, we showed the following:

Proposition 4.2 (Proposition 30.78 in [6])**.** *For any g_0, n, K, there exists $g_1 > g_0$ and a filtered $A_{n,K}$-structure $\mathfrak{m}_{k,\beta}$ on $C_{(g_1)}(L) \otimes \Lambda_{0,nov}$ such that*

$$\mathfrak{m}_{k,\beta}(P_1,\dots,P_k) = \mathfrak{m}^{geo}_{k,\beta}(P_1,\dots,P_k),$$

if $P_i \in \chi_{(g_0)}(L)$.

Combining Theorem 2.4 and Proposition 4.2, we can construct a filtered A_∞-algebra associated to L, for details see [6]. Hence we obtain

Theorem 4.3 (Theorem 10.11 in [6])**.** *Let L be a relatively spin Lagrangian submanifold. Then there exist a countably generated subcomplex $C(L)$ of the singular chain complex and a filtered A_∞-algebra structure on $C(L) \otimes \Lambda_{0,nov}$.*

We also proved that the homotopy type of the filtered A_∞-algebra is unique.

Applying the construction of canonical models in section 3, we obtain a filtered A_∞-algebra structure on $\mathrm{H}(L) \otimes \Lambda_{0,nov}$.

Let (L_0, L_1) be a relative spin pair of Lagrangian submanifolds. Assume that L_0 and L_1 intersect transversely. Then we have the following:

Theorem 4.4. *Let $D^\bullet$ be a free $\Lambda_{0,nov}$-module generated by $L_0 \cap L_1$. Then there exists a filtered A_∞-bimodule structure over filtered A_∞-algebras associated to L_i, $i = 0, 1$.*

5. Canonical models and Morse complexes

In this section, we apply Theorem 3.3 and reduce the filtered A_∞-structure on $C^\bullet(L) \otimes \Lambda_{0,nov}$ to the Morse complex $CM^\bullet(f) \otimes \Lambda_{0,nov}$.

We pick a specific Morse function as follows. Choose and fix a triangulation $\mathfrak{T}$ of L. We may assume that the triangulation is sufficiently fine by taking subdivision. Pick a Morse function $f : L \to \mathbb{R}$ with the following property. Critical points of f are in one-to-one correspondence with barycenters of simplices. Moreover, the Morse index of a critical point is equal to the dimension of the corresponding simplex. Then we can take a gradient-like vector field X such that the unstable manifold $W^u(p)$ at each critical point p is the interior of the corresponding simplex. Denote by $\{\rho_t\}$ the flow generated by X. (The function f increases along the orbits of $\{\rho_t\}$.)

Now we prove the following:

Theorem 5.1. *Let L be a relatively spin Lagrangian submanifold in a closed symplectic manifold (M,ω) and f a Morse function on L as above. Then Morse complex $CM^*(f) \otimes \Lambda_{0,nov}$ carries a structure of a filtered A_∞-algebra, which is homotopy equivalent to the filtered A_∞-algebra associated to L constructed in* [6].

The proof occupies the rest of this section. We explain how to choose $\chi_g(L)$ in section 4. Firstly, we choose and fix a linear order on the set of vertices in $\mathfrak{T}$. Then we regard each $T_i \in \mathfrak{T}$ as a singular simplex by the affine parametrization $\sigma_i : \Delta_{k_i} \to T_i$ respecting the order of the vertices. In particular, all simplices are oriented, hence the unstable manifolds $W^u(p)$. For our construction, we have to start with the following set of singular simplices. Set $\chi_{\mathfrak{T}}(L) = \{\sigma_i\}$ and identify the Morse complex $CM^\bullet(f)$ with $C_{\mathfrak{T}}(L)$, which is a subcomplex of the singular chain complex of L generated by $\chi_{\mathfrak{T}}(L)$. Note that $\chi_{\mathfrak{T}}(L)$ satisfies Conditions 1 and 2 given in section 4.

We define $\chi_g(L) \supset \chi_{\mathfrak{T}}(L)$ in an inductive way as follows. For $g = -1$, we set $\chi_{-1}(L) = \chi_{\mathfrak{T}}(L)$. For $g = 0, 1, \ldots,$ suppose that we constructed $\chi_{g'}(L)$, $g' < g$.

We can choose the perturbations $\mathfrak{s}_{\mathfrak{d},k,\beta,\vec{P}}$ in Proposition 4.1 with the following property.

Each face τ of any simplex in the triangulation of

$$ev_{0*}\big(\mathcal{M}_{k+1}^{\mathrm{main},\mathfrak{d}}(\beta; P_1, \ldots, P_k)^{\mathfrak{s}_{\mathfrak{d},k,\beta,\vec{P}}}\big)$$

with $g = \| (\mathfrak{d}, \beta) \|$ is transversal to the stable manifold $W^s(p)$ at any $p \in \mathrm{Crit}(f)$. Moreover, for each τ of dimension at most $\dim L$, there exists at most one $p = p(\tau) \in \mathrm{Crit}(f)$ such that the stable submanifold $W^s(p)$ is of complementary dimension to τ and $W^s(p)$ and τ intersect at a unique point. Denote by $T(p) \in \mathfrak{T}$ the simplex containing p. Let $\chi_g^\circ(L)$ be the set of these singular simplices τ.

We have to add $\chi_g^\circ(L)$ to previous $\bigcup_{g'<g} \chi_{g'}(L)$. In order to guarantee Condition 2, we further add the following singular simplices to $\chi_g^\circ(L)$ and obtain $\chi_g(L)$. Denote by $\sigma^\tau \in \chi_{\mathfrak{T}}$ the singular simplex corresponding to $T(p(\tau))$. Define $\Pi(\tau) = \epsilon\sigma^\tau$, where $\epsilon = \pm 1$ is given by the following equation.

$$\tau \cap W^s(p(\tau)) = \epsilon W^u(p(\tau)) \cap W^s(p(\tau)),$$

if there exists a unique stable manifold $W^s(p(\tau))$, which intersects τ transversely at a unique point. Otherwise, we define $\Pi(\tau) = 0$. In particular, if $\tau > \dim L$, $\Pi(\tau) = 0$. For each τ as above, we will find a singular chain $G(\tau)$ such that

$$\Pi(\tau) - \tau = \overline{\mathfrak{m}}_1 G(\tau) + G(\overline{\mathfrak{m}}_1 \tau),$$

where $\overline{\mathfrak{m}}_1 = (-1)^{\dim L}\partial$. We can find such $G(\tau)$ by induction on dimension of τ. In our case, we construct $G(\tau)$ using the gradient-like flow $\{\rho_t\}$. Set

$$\mathfrak{r}(\mathrm{Im}\tau) = \bigcup_{t \le 0} \rho_t(\mathrm{Im}\tau).$$

By the choice of our perturbations above, the closure of $\mathfrak{r}(\mathrm{Im}\tau)$ can be triangulated in a compatible way with τ and $\Pi(\tau)$. Pick such a triangulation and then define $G(\tau)$ the corresponding singular chain. For the chain $G(\tau)$, we define $G(G(\tau)) = 0$.

Note that $\Pi : C_{(g)}(L) \to C_{(0)}(L)$ and $G : C_{(g)}(L) \to C_{(g)}(L)$ satisfy the conditions in Lemma 3.2, hence $C_{(g)}(L)$ satisfies Condition 2. Therefore we can apply Theorem 3.3 to reduce the filtered A_∞-structure on $C^\bullet(L; \Lambda_{0,nov})$ to $CM^\bullet(f) \otimes \Lambda_{0,nov}$ and obtain a filtered A_∞-algebra $(CM^\bullet(f) \otimes \Lambda_{0,nov}, \{\mathfrak{m}'_k\})$, which is homotopy equivalent to $(C_{(g)}(L) \otimes \Lambda_{0,nov}, \{\mathfrak{m}_k\})$. Theorem 5.1 is proved.

In the proof of Theorem 3.3, we constructed the operator $\mathfrak{m}'_k$ from $\mathfrak{m}_\Gamma$, $\Gamma \in G_{k+1}^+$. The geometric meaning of $\mathfrak{m}_\Gamma$ is as follows. Recall that $G(\tau)$ assigns the closure of the union of flow lines arriving at τ. We assign G to the interior edges. The interior vertices correspond to J-holomorphic discs, more precisely, bordered stable maps

of genus 0. In order to describe the operation $\mathfrak{m}_\Gamma$, we need only rigid configuration of $\tau_i \in \chi_{\mathfrak{T}}(L)$ (the barycenters of τ_i are inputs), J-holomorphic discs, (broken) negative flow lines of X and $W^s(q)$ (q is the output). We choose the perturbation $\mathfrak{s}$ generically so that the moduli spaces of holomorphic discs and the flow $\{\rho_t\}$ are in general position so that the inner edges correspond to negative flow lines of X. Hence the $\mathfrak{m}_\Gamma$ is defined by using the configuration of pseudo-holomorphic discs and Morse negative gradient trajectories according to the decorated tree $\Gamma \in \cup_k G^+_{k+1}$.

For a decorated tree $\Gamma \in G^+_{k+1}$, each edge is oriented in the direction from the k input vertices to the root vertex. We denote by $v^\pm(e)$ the vertices such that e is an oriented edge from $v^-(e)$ to $v^+(e)$. Consider the moduli space $\mathcal{M}_\Gamma(h; p_1, \dots . p_k, q)$ consisting of the configuration of the following

- for each interior vertex $v \in \Gamma$, a bordered stable map u_v representing the class $\beta_{\eta(v)}$ with $\ell(v)$ boundary marked points, where $\ell(v)$ is the valency of v, (we denote by $p(e, v)$ the marked point corresponding to the edge e attached to v)
- the i-th input edge e_i corresponds to a broken negative gradient flow line γ_i starting from the critical point p_i to $u_{v^+(e_i)}(p(e_i, v^+(e_i)))$,
- the output edge corresponds to a broken negative gradient flow line γ_0 from $u_{v^-(e_0)}(p(e_0, v^-(e_0)))$ ending at the critical point q,
- an interior edge e corresponds to a broken negative gradient flow line γ_e from $u_{v^-(e)}(p(e, v^-(e)))$ to $u_{v^+(e)}(p(e, v^+(e)))$.

Counting the weighted order of the moduli spaces of vitual dimension 0, we get

$$\mathfrak{m}_\Gamma(p_1 \otimes \cdots \otimes p_k) = \sum_q \#\mathcal{M}_\Gamma(h; p_1, \dots, p_k, q) \cdot e^{\sum_v \mu(\beta_{\eta(v)})/2} q$$

and

$$\mathfrak{m}_k = \sum_{\Gamma \in G^+_{k+1}} T^{E(\Gamma)} \mathfrak{m}_\Gamma.$$

For example, we obtain the configuration as in Figure 6 associated to the decorated planar tree Γ with inputs $T(p), T(p'), T(p'') \in \chi_{\mathfrak{T}}(L)$ as in Figure 5.

This is essentially the configuration introduced in [4]. Note that the first named author [4] took multiple Morse functions to achieve transversality. Here we use one Morse function and apply the argument in section 3 to squeeze the filtered A_∞-algebra structure to the Morse complex. We emphasize that this becomes possible only after working out the chain level intersection theory in detail, which we explained in section 4. To find an appropriate perturbation of $\mathcal{M}_\Gamma(h; p_1, \dots, p_k, q)$ directly without using the argument in section 4 (or section 30 in [6]) seems extremely difficult.

The use of multiple Morse functions enables to construct the topological (or partial) filterd A_∞-category of Morse functions on L in the case that $\mathfrak{m}_0 = 0$. Note that, in a topological (or partial) filtered A_∞-category $\mathcal{A}$, the set $Ob_{\mathcal{A}}$ of objects is a topological space and the set $Mor_{\mathcal{A}}(a, b)$ of morphisms is defined for (a, b) in an open dense subset of $Ob_{\mathcal{A}} \times Ob_{\mathcal{A}}$. When $\mathcal{A}$ is a filtered A_∞-category, each object a is equipped with the filtered A_∞-algebra $Mor_{\mathcal{A}}(a, a)$. In our case, the filtered A_∞-algebra on Morse complex $CM^\bullet(f)$ corresponds to the filtered A_∞-algebra associated to the object f. Note that, in the construction of this paper and in Theorem 5.1, we do not need to assume that $\mathfrak{m}_0 = 0$ in our construction.

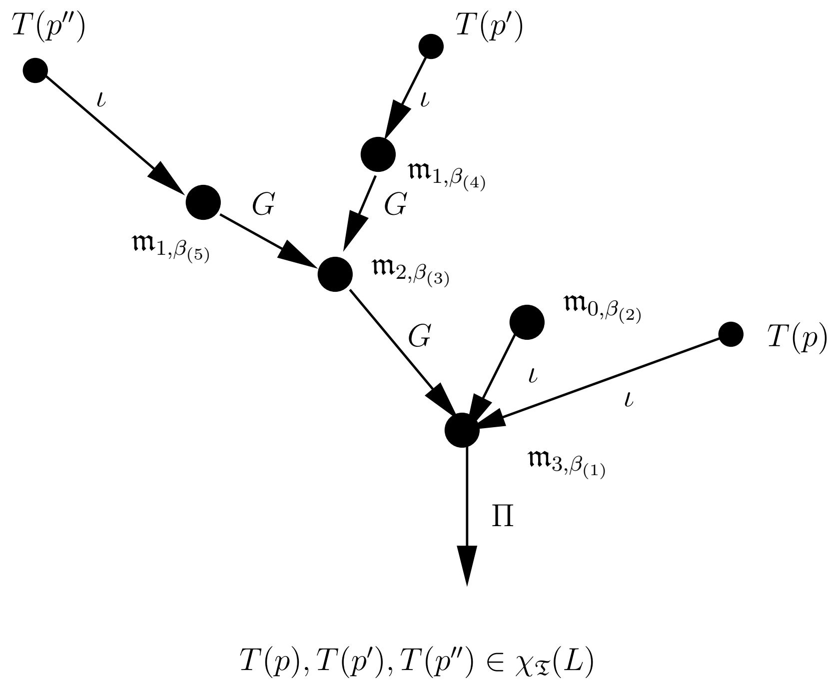

FIGURE 5.

For a relative spin pair (L_0, L_1) of Lagrangian submanifolds, which intersect transversely, we obtain the filtered A_∞-bimodule over the filtered A_∞-algebras on $CM^\bullet(f_i) \otimes \Lambda_{0,nov}$, where $f_i : L_i \to \mathbb{R}$, $i = 0, 1$, are Morse functions. When L_0 and L_1 are of clean intersection, there exists a certain local system Θ on $L_0 \cap L_1$ and we can reduce the filtered A_∞-bimodule structure on $C^\bullet(L_0 \cap L_1; \Theta) \otimes \Lambda_{0,nov}$ to $CM^\bullet(h; \Theta) \otimes \Lambda_{0,nov}$ over the filtered A_∞-algebras on $CM^\bullet(f_i) \otimes \Lambda_{0,nov}$. Here h is a Morse function on $L_0 \cap L_1$, which may be disconnected with various dimensions. For the canonical models of filtered A_∞-bimodules, see [6].

Acknowledgement. We thank Otto van Koert for his kind instruction of making figures in this article.

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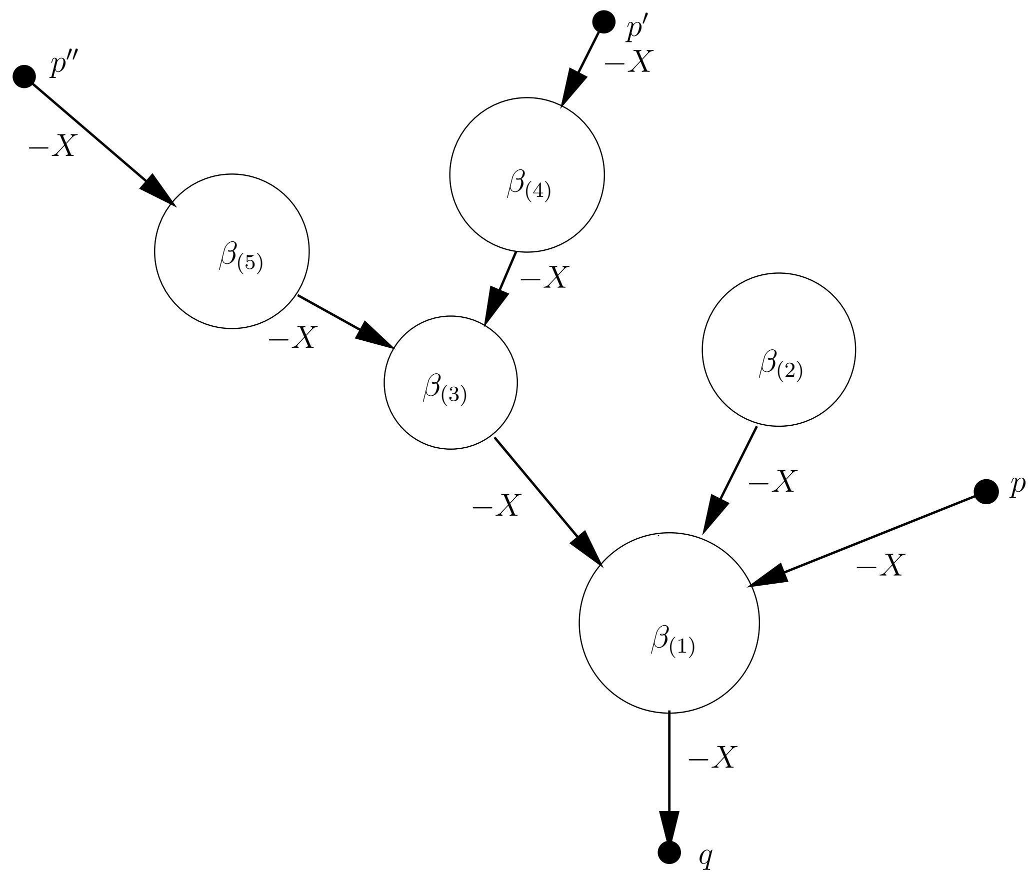

p, p', p'', q are critical points of the Morse function.

Figure 6.

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K. Fukaya: Department of Mathematics, Kyoto University, Kyoto, 606-8502, Japan

Y.-G. Oh: Department of Mathematics, University of Wisconsin, Madison, WI 53706, U.S.A. and Korea Institute for Advanced Study, Seoul, Korea

H. Ohta: Graduate School of Mathematics, Nagoya University, Nagoya, 464-8602, Japan

K. Ono: Department of Mathematics, Hokkaido University, Sapporo, 060-0810, Japan